Some estimates for Banach space norms in the von Neumann algebras associated with the Berezin's quantization of compact Riemann surfaces

by Florin Rădulescu[1,2]

Abstract. *Let Γ be any cocompact, discrete subgroup of $PSL(2,\mathbb{R})$. In this paper we find estimates for the predual and the uniform Banach space norms in the von Neumann algebras associated with the Berezin' s quantization of a compact Riemann surface $\mathbb{D}/\Gamma$. As a corollary, for large values of the deformation parameter $1/h$, these von Neumann algebras are isomorphic.*

Using the results in [AS], [AC], [GHJ] on the von Neumann dimension of the Hilbert spaces in the discrete series of unitary representations of $PSL(2,\mathbb{R})$, as left modules over Γ we deduce that the fundamental group ([MvN]) of the von Neumann $\mathcal{L}(\Gamma)$ contains the positive rational numbers. Equivalently, this proves that the algebras $\mathcal{L}(\Gamma)\otimes M_n(\mathbb{C})$, $n\in\mathbb{N}$ are mutually isomorphic.

In this paper we will find some estimates for the uniform (Banach) norm in the von Neumann algebras in the Berezin's deformation quantization of compact Riemann surfaces. We then use these estimates together with the results in ([Ra1]) to deduce that the von Neumann algebras in the Berezin's deformation, are for large values of the deformation parameter ($r = 1/h$), mutually isomorphic. The results in the papers ([AS], [Co2], [GHJ]) prove that the Hilbert spaces, associated with the quantization, are finite, left modules over the type II_1 factor $\mathcal{L}(\Gamma)$.

Consequently this implies that the von Neumann algebras in the deformation quantization are stably (Morita) equivalent with $\mathcal{L}(\Gamma)$. By using the isomorphism result proved in this paper one deduces that the algebras $\mathcal{L}(\Gamma)\otimes M_n(\mathbb{C})$, $n\in\mathbb{N}$ are mutually isomorphic.

This in turn, because of the existing results in the literature, may be used to show that $\mathcal{L}(\Gamma)$ has (non-irreducible) subfactors having non-integer indices (by using [Jo1]) or to show that the algebra $\mathcal{L}(\Gamma)$ is singly generated (by using [To]). The above result implies, (by using [Co1]), that there are type III_λ factors whose core is $\mathcal{L}(\Gamma)\otimes B(H)$ for a separable Hilbert space H. Using [HP], one obtains that there exists no bounded projection from $B(H)$ onto the Banach space subjacent to $\mathcal{L}(\Gamma)$.

[1]Permanent Address: Department of Mathematics, The University of Iowa, Iowa City, IA 52246, U.S.A

[2]Member of the Institute of Mathematics, Romanian Academy, Str. Academiei 14, Bucharest, Romania

Uniform norm for convolutors in von Neumann algebras of discrete groups have been first computed by Akemann and Ostrand in ([AO]). Estimates for the uniform norm in free group algebras have been determined by Haagerup in [Ha] and then used by him to find a non-nuclear C^*-algebra with the approximation property. Some of these estimates have been generalized by Jollisaint ([Jo]) for a larger class of groups (including cocompact groups and some of Gromov's hyperbolic groups). Some of these results are used in the papers of Connes and Moscovici ([Co]) on the local index formulae. We refer to [Pi] (and the references therein) for the connections between the computation of such norms and recent developments in the theory of Banach spaces.

The computation for norms of convolutors in the von Neumann algebra of a free group are part of the Voiculescu's non-commutative probability theory ([Vo3], [Vo4]. One very important consequence of the Voiculescu's theory of random matrices, as asymptotic models for free group factors, is that the fundamental group of the von Neumann algebra of a free group with infinitely many generators contains the rational numbers ([Vo1]).

In this paper we consider the von Neumann algebras of the Berezin's quantization of a Riemann surface realized as $\mathbb{D}/\Gamma$ for a cocompact subgroup Γ of $SU(1,1)$. As we mentioned above, the algebras in this deformation are stably (Morita) equivalent with $\mathcal{L}(\Gamma)$. In [Ra1] we proved that any element in the von Neumann algebras in the deformation is represented by a kernel, a function on $\mathbb{D}\times\mathbb{D}$, $k=k(\overline{z},\zeta)$, which is analytic in the second variable and antianalytic in the first. Moreover the kernel is Γ invariant, that is $k(\overline{\gamma z},\gamma\zeta)=k(\overline{z},\zeta)$, $\gamma\in\Gamma$ and $z,\zeta\in\mathbb{D}$. We will prove that the uniform norm of the element represented by k is equivalent to the following quantity (r is the reciprocal of the Planck's constant h):

$$\max\,\{\sup_{z\in\mathbb{D}}\int_{\mathbb{D}}|k(\overline{z},\zeta)|(d(z,\zeta))^r\mathrm{d}\lambda_0(\zeta),\sup_{\zeta\in\mathbb{D}}\int_{\mathbb{D}}|k(\overline{z},\zeta)|(d(z,\zeta))^r\mathrm{d}\lambda_0(z)\}.$$

By using this estimate and the results in ([Ra1]) we prove that for any cocompact, fuchsian group subgroup Γ of the group $PSL(2,\mathbb{R})$ (canonically identified with the group $SU(1,1)$), the associated von Neumann algebra $\mathcal{L}(\Gamma)=\overline{\mathbb{C}(\Gamma)}^w\subseteq B(l^2(\Gamma))$ has the property that the fundamental group

$$\mathcal{F}(\mathcal{L}(\Gamma))=\{t\mid e(\mathcal{L}(\Gamma)\otimes M_n(\mathbb{C}))e\simeq\mathcal{L}(\Gamma),\ \text{for some projection } e \text{ of trace } t\leq n\}$$

contains $\mathbb{Q}_+\backslash\{0\}$. Equivalently, this proves that the algebras $\mathcal{L}(\Gamma)\otimes M_n(\mathbb{C})$, $n\in\mathbb{N}$, are mutually isomorphic.

As it was pointed out in [HV] (see also the references therein) the type II_1 factors associated with cocompact groups in $PSL(2,\mathbb{R})$ have the non-Γ-property of Murray and von Neumann. The property which we prove in this paper, for the von Neumann algebra of a cocompact group in $PSL(2,\mathbb{R})$, is similar to the corresponding property of the von Neumann algebra of a free group with infinitely many generators. For this last group, it was a breakthrough discovery of Voiculescu ([Vo2]), based on the random matrix model, that the fundamental group $\mathcal{F}(\mathcal{L}(F_\infty))$ contains the positive rationals (in fact, as proved in [Ra4], this model may be used to show that the fundamental group $\mathcal{F}(\mathcal{L}(F_\infty))$ is $\mathbb{R}_+\backslash\{0\}$).

The similar problem for free groups with finitely many generators is widely open. Based on Voiculescu's random matrix model for free groups F_N with finitely many generators, it was proved independently in ([Dy], [Ra1]) that $\mathcal{F}(\mathcal{L}(F_N))$ is either $\mathbb{R}_+\backslash\{0\}$ or either $\{1\}$, independently on the natural number N. The first situation would occur if and only if one would have a positive answer to the von Neumann-Kadison-Sakai question on the isomorphism of the free group algebras $\mathcal{L}(F_N)$.

We finally note that the only other type II_1 factors (except for the hyperfinite factor) for which one has some knowledge about the fundamental group, are the algebras associated with groups with the property T. For this algebras, by a remarkable result of A. Connes, ([Co3]), we know that the fundamental group is almost countable (see also [Po] for the recent construction of a different type II_1 factor with the same property).

Acknowledgment. *During the elaboration of this paper, I have greatly benefited from disscutions with C. Frohman, U. Haagerup, C. T. McMullen, G. Pisier, R. Curto and D. Voiculescu.*

Definitions and outline of proofs

The proof of our main result is based on some estimates for the Banach space norms on the von Neumann algebras associated with the $\Gamma-$ equivariant, Berezin's quantization of the unit disk ([Ra2]) or, in other words, for the algebras associated with the quantization deformation ([Be]) of the compact Riemann surface $\mathbb{D}/\Gamma$.

In this deformation quantization, the associated von Neumann are type II_1 factors ([Ra2], see also [Ri] for deformations quantization with similar behavior). We prove that the von Neumann algebras corresponding to different values of the deformation parameter h are all isomorphic, if $x-1)h$ is sufficiently large (depending

on the exponent of convergence ([Be], [Pa]) of the group Γ). We refer to [En] for the computations regarding the asymptotics of the Berezin's deformation quantization for such domains (see also [BMS], [KL], [BC]).

Let λ_r be the measure on $\mathbb{D}$ defined by $\mathrm{d}\lambda_r(z) = (1-|z|^2)^{r-2}\mathrm{d}z\mathrm{d}\overline{z}$. Hence λ_0 is the $PSL(2,\mathbb{R})$-invariant measure on $\mathbb{D}$. Let $(\pi_r)_{r>1}$ be the (continuous) series of projective, unitary representations of $PSL(2,\mathbb{R})$, (identified with $SU(1,1)$), on the Hilbert space $H_r = H^2(\mathbb{D}, \mathrm{d}\lambda_r)$ ([Sa]). For M a type II_1 factor with normalized trace τ and for any t in $[0,1]$, following ([MvN), let the reduced algebra M_t be the isomorphism class of the type II_1 factor eMe, where e is any selfadjoint idempotent of trace t in M.

Let $M_n(\mathbb{C})$ carry the canonical (non-normalized) trace. The definition for M_t definition also makes sense for any $t > 1$ if we replace, from the beginning, the algebra M by $M \otimes M_n(\mathbb{C})$, where n is any integer bigger then t. The isomorphism class of M_t is independent of the choices made so far ([MvN]). In particular the fundamental group $\mathcal{F}(M)$ is the multiplicative group $\{t | M_t \cong M\}$.

In [Ra2] we proved that for each $h = 1/r > 0$ there exists a suitable vector space $\mathcal{V}_h$ consisting of smooth, $\Gamma-$invariant functions on $\mathbb{D}$ (or simply, consisting of smooth functions on $\mathbb{D}/\Gamma$), so that $\mathcal{V}_h$ is closed under conjugation and under the Berezin product $*_h$. Moreover if we endow $\mathcal{V}_h$ with the trace τ given by the integral over a fundamental domain F of Γ in $\mathbb{D}$, then, (by the Gelfand-Naimark -Segal construction), we obtain a type II_1 factor $\mathcal{A}_r$ that coincides with the commutant $\{\pi_r(\Gamma)\}'$ of the image of Γ in $B(H_r)$ through the projective, unitary representation π_r of $PSL(2,\mathbb{R})$ into $B(H_r)$. By cov Γ we denote the covolume of Γ. For integer r, (or if Γ is the (non-cocompact) group $PSL(2,\mathbb{Z})$), it is known, (see [Co3], [GHJ], [Ra2]), that the algebras $\mathcal{A}_r$ are isomorphic to the reduced algebra $\mathcal{L}(\Gamma)_{[(r-1)(\mathrm{cov}\ \Gamma)/\pi]}$.

For z, ζ in $\mathbb{D}$ let $d(z,\zeta) = (1-|z|^2)^{1/2}(1-|\zeta|^2)^{1/2}|1-\overline{z}\zeta|^{-1}$. This is the square root of the hyperbolic cosine of the hyperbolic distance between z and ζ in $\mathbb{D}$. For any z in $\mathbb{D}$ let e^r_z be the vector in H_r which corresponds to the evaluation at z. Let A be a bounded, linear operator on H_r. Let $\langle \cdot, \cdot \rangle_r$ be the scalar product on H_r. Recall ([Be]) that the Berezin's (contravariant) symbol of A is a function $\hat{A}$ on $\mathbb{D}\times\mathbb{D}$, antianalytic in the first variable and analytic in the second, computed by the formula

$$\hat{A}(\overline{z}, \zeta) = \langle Ae^r_z, e^r_\zeta \rangle (\langle e^r_z, e^r_\zeta \rangle)^{-1}$$

If A commutes with $\pi_r(\Gamma)$ then the symbol has the following invariance property:

$$\hat{A}(\gamma\overline{z}, \gamma\zeta) = \hat{A}(\overline{z}, \zeta),$$

for all z, ζ in $\mathbb{D}$ and for all γ in Γ.

In [Ra2] we introduced the following norm (which is stronger then the uniform norm) on a weakly dense subalgebra of $B(H_r)$. The definition of the norm, for A in $B(H_r)$, is given by

$$||A||_{\lambda,r} = \max \{\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |\hat{A}(\overline{z}, \zeta)|(d(z,\zeta))^r \mathrm{d}\lambda_0(\zeta), \sup_{\zeta\in\mathbb{D}} \int_{\mathbb{D}} |\hat{A}(\overline{z}, \zeta)|(d(z,\zeta))^r \mathrm{d}\lambda_0(z)\}.$$

Let $\widehat{B(H_r)}$ be the set of all bounded operators on H_r whose $||\cdot||_{\lambda,r}$ norm is finite and let $\hat{\mathcal{A}}_r$ be $\mathcal{A}_r \cap \widehat{B(H_r)}$. In the same paper ([Ra2]), by using the explicit formulae for the Berezin's multiplication rule $*_h$ we determined an explicit formula for the cyclic, two cocycle ψ_r, canonically associated with the deformation (see also [CFS], [CM], [RN]). The formula for ψ_r proved that ψ_r lives on the algebra $\hat{\mathcal{A}}_r$, and that the following estimate holds

$$(1) \qquad |\psi_r(A, B, C)| \leq \mathrm{const}_r ||A||_{\lambda,r} ||B||_2 ||C||_2, \text{ for all } A, B, C \in \hat{\mathcal{A}}_r.$$

In general the cyclic cohomology class ([Co]) of the cocycle ψ_r represents an obstruction for the different products $*_h$ to define isomorphic algebras. The construction of ψ_r may be used (see [Ra2]) to prove that the bounded cohomology group $H^2_{\mathrm{bounded}}(\Gamma, \mathbb{Z})$, ([Gr], [Gh]), is nontrivial for any discrete subgroup Γ of $PSL(2, \mathbb{R})$ having finite covolume.

If the cocycle ψ_r is bounded, by the uniform norm $||\cdot||_{\infty,r}$ on $\mathcal{A}_r$ replacing the norm $||\cdot||_{\lambda,r}$ in the equality (1), then standard techniques ([SinS], [CES], [PR]) in the cohomology theory of von Neumann algebras are used to show ([Ra2]) that ψ_r is the boundary of a bounded cycle ϕ_r. Hence there exists a bounded, antisymmetric, linear operator X_r on $L^2(\mathcal{A}_r)$ so that for all A, B in $L^2(\mathcal{A}_r)$ we have that $\phi_r(A, B) = \langle X_r(A), B\rangle$. The evolution operators on $L^2(\mathcal{A}_r)$, corresponding to the non-autonomous differential equation associated with X_r will then implement an isomorphism between the algebras $\mathcal{A}_r$ for different r's ([Ra2]).

In this paper, we show that for a cocompact, discrete subgroup Γ of $PSL(2, \mathbb{R})$, the norms $||\cdot||_{\lambda,r}$ and $||\cdot||_{\infty,r}$ are equivalent on $\mathcal{A}_r \cap \{\pi_r(\Gamma)\}'$. Thus, as described

above, it follows that the algebras $\mathcal{A}_r$, associated with the Berezin's deformation quantization of $\mathbb{D}\backslash\Gamma$, are mutually isomorphic. By ([AS], [Co2], [GHJ]) the algebras $\mathcal{A}_r = \{\pi_r(\Gamma)\}'$ are isomorphic, for integers $r \geq 2$, to $\mathcal{L}(\Gamma)_{(r-1)(\mathrm{cov}\ \Gamma)/\pi)}$.

Note that, (by [Ra2]), this holds in fact for any $r > 1$, if the cocycle coming from the projective, unitary representation $\pi_r|_\Gamma$ is trivial in $H^2(\Gamma, \mathbb{T})$ (which happens if e. g. Γ is the (non-cocompact) subgroup $PSL(2,\mathbb{Z})$. Hence, for sufficiently large integers n, m, the algebras $\mathcal{L}(\Gamma)_{[(n-1)(\mathrm{cov}\ \Gamma)/\pi]}$ and $\mathcal{L}(\Gamma)_{[(m-1)(\mathrm{cov}\ \Gamma)/\pi]}$ are isomorphic. This implies that the fundamental group of $\mathcal{L}(\Gamma)$ contains the positive rational numbers.

Note that if the conjecture in [HV] asserting that the von Neumann algebra of a cocompact, discrete subgroup of $PSL(2,\mathbb{R})$ is isomorphic to the algebra of free group whose fractional "number of generators" ([Dy], [Ra1], [Vo1])) depends on the covolume of Γ, then it would follow (by [Vo2], [Dy], [Ra1]) that the question (von -Neumann-Kadison-Sakai, [Ka], [Sa]) on the isomorphism of the algebras $\mathcal{L}(F_N)$ would have an affirmative solution. Alternatively this could happen if one could extend the methods in this paper to non-cocompact groups like $PSL(2,\mathbb{Z})$ or to the discrete Hecke subgroups.

For z, ζ in $\mathbb{D}$ the function $d(z,\zeta) = (1-|z|^2)^{1/2}(1-|\zeta|^2)^{1/2}|1-\overline{z}\zeta|^{-1}$ is the square root of the hyperbolic cosines of the hyperbolic distance between z and ζ (see e. g. [Pa]). Denote by K_r the symmetric, Γ-equivariant kernel on $\mathbb{D}$ defined by

$$K_r(z,\eta) = \sum_{\gamma\in\Gamma} d(\gamma\eta, z)^r, z, \eta \in \mathbb{D}.$$

It is well known that the series defining K_r is uniformly convergent on compact subsets of $\mathbb{D}$ if r is bigger then the double of the exponent of convergence of the group Γ ([Be], [Le], [Pa]). This types of kernels appear in the Selberg trace formula ([Se]).

Let F be any fundamental domain for Γ acting on $\mathbb{D}$. Recall that the trace on $\mathcal{A}_r$ is defined by the formula $\tau_{\mathcal{A}_r}(A) = \tau(A) = (\lambda_0(F))^{-1}\int_F \hat{A}(\overline{z}, z)\lambda_0(z)$ for any A in $\mathcal{A}_r$ with Berezin symbol $\hat{A}$. The formula for the product of two elements A, B in $\mathcal{A}_r$ is computed out of the symbols $\hat{A}, \hat{B}$ as

$$(\hat{A} * \hat{B})(\overline{z}, \zeta) = c_r \int \frac{\hat{A}(\overline{z}, \eta)}{(1-\overline{z}\eta)^r} \frac{\hat{B}(\overline{\eta}, \zeta)}{(1-\overline{\eta}\zeta)^r} d\lambda_0(\eta), z, \zeta \in \mathbb{D}.$$

Hence the Hilbert space $L^2(\mathcal{A}_r,\tau)$ associated (via the Gelfand-Naimark-Segal) construction to the trace τ on the type II_1 factor $\mathcal{A}_r=\{\pi_r(\Gamma)\}'$ is identified with the Hilbert space of functions $k=k(\overline{z},\eta),\gamma\in\Gamma$ on $\mathbb{D}\times\mathbb{D}$ that are antianalytic in the first variable, analytic in the second which are $\Gamma-$invariant $(k(\overline{\gamma z},\gamma\zeta)=k(\overline{z},\zeta))$. The Hilbert norm is given by the formula:

$$||k||_{2,r}^2=\int\int_{\mathbb{D}\times F}|k(\overline{z},\eta)|^2(d(z,\eta))^{2r}\mathrm{d}\lambda_0(z)\mathrm{d}\lambda_0(\zeta).$$

Here $\mathbb{D}\times F$ could be replaced by any fundamental domain for the diagonal action of Γ on $\mathbb{D}\times\mathbb{D}$.

The results

In the next lemma we determine the precise formula for the point evaluation vectors in the Hilbert space associated to the deformation quantization. As we pointed out above this may be identified with a Hilbert space of square summable analytic functions and hence it contains evaluation vectors.

Lemma. 1. *Let Γ be a cocompact subgroup of $PSL(2,\mathbb{R})$. For z,ζ in $\mathbb{D}$ and for every r bigger then the double of the exponent of convergence of Γ let $e^r_{\overline{z},\zeta}=e^r_{\overline{z},\zeta}(\overline{\eta_1},\eta_2),\eta_1,\eta_2$ in $\mathbb{D}$ be the function on $\mathbb{D}^2$, antianalytic in the first variable, analytic in the second defined by the formula:*

$$e^r_{\overline{z},\zeta}(\overline{\eta_1},\eta_2)=\frac{r-1}{\pi}\sum_\gamma\frac{(1-\overline{\gamma\eta_1}\gamma\eta_2)^r(1-\overline{z}\zeta)^r}{(1-(\overline{z})(\gamma\eta_2))^r(1-\overline{\gamma\eta_1}\zeta)^r}$$

$$=\frac{r-1}{\pi}\sum_\gamma\frac{(1-\overline{\eta_1}\eta_2)^r(1-\overline{\gamma z}\gamma\zeta)^r}{(1-(\overline{\gamma z})\eta_2)^r(1-(\overline{\eta_1})\gamma\zeta)^r},\eta_1,\eta_2\in\mathbb{D}.$$

Then $e^r_{\overline{z},\zeta}$ is the evaluation vector at $z,\zeta\in\mathbb{D}$ on $L^2(\mathcal{A}_r,\tau)$, that is $\tau(Ae^r_{\overline{z},\zeta})=\hat{A}(\overline{z},\zeta)$ for all A in $L^2(\mathcal{A}_r,\tau)$. Moreover $e^r_{\overline{z},\zeta}$ belongs to $\hat{\mathcal{A}}_r\subseteq\mathcal{A}_r$.

Proof. We first prove that the series defining $e^r_{\overline{z},\zeta}$ is uniformly convergent on compact subsets in $\mathbb{D}\times\mathbb{D}$. This will follow automatically from the computations showing that $e^r_{\overline{z},\zeta}$ belongs to $\hat{\mathcal{A}}_r$. We need to estimate

$$\sup_{\eta_1\in\mathbb{D}}\int_{\mathbb{D}}\sum\frac{|1-\overline{\eta_1}\eta_2|^r|1-\overline{\gamma z}\gamma\zeta|^r}{|1-\overline{\gamma z}\eta_2|^r|1-\overline{\eta_1}\gamma\zeta|^r}(d(\eta_1,\eta_2))^r\mathrm{d}\lambda_0(\eta_2)$$

$$= \sup_{\eta_1 \in \mathbb{D}} \sum_\gamma \frac{|1 - \overline{\gamma z} \gamma \zeta|^r}{|1 - \overline{\eta_1} \gamma \zeta|^r} (1 - |\eta_1|^2)^{r/2} \int_{\mathbb{D}} \frac{1}{|1 - \overline{\gamma z} \eta_2|^r} \lambda_{r/2}(\eta_2)$$

$$= \sup_{\eta_1 \in \mathbb{D}} \sum_\gamma \frac{|1 - \overline{\gamma z} \gamma \zeta|^r (1 - |\eta_1|^2)^{r/2}}{|1 - \overline{\eta_1} \gamma \zeta|^r (1 - |\gamma z|^2)^{r/2}}$$

$$= \sup_{\eta_1 \in \mathbb{D}} \sum_\gamma \frac{|1 - \overline{\gamma z} \gamma \zeta|^r}{(1 - |\gamma z|^2)^{r/2} (1 - |\gamma \zeta|^2)^{r/2}} \frac{(1 - |\eta_1|^2)^{r/2} (1 - |\gamma \zeta|^2)^{r/2}}{|1 - \overline{\eta_1} \gamma \zeta|^r}$$

$$= \sup_{\eta_1 \in \mathbb{D}} (d(z, \zeta))^{-r} \sum_\gamma \frac{(1 - |\eta_1|^2)^{r/2} (1 - |\gamma \zeta|^2)^{r/2}}{|1 - \overline{\eta_1} \gamma \zeta|^r}$$

$$= \sup_{\eta_1 \in \mathbb{D}} (d(z, \zeta))^{-r} K_r(\zeta, \eta_1) \leq M_r (d(z, \zeta))^{-r}.$$

Hence $e^r_{\overline{z}, \zeta}$ belongs to $\hat{\mathcal{A}}_r$ and

$$||e^r_{\overline{z}, \zeta}||_{\lambda, r} \leq M_r (d(z, \zeta))^{-r}.$$

The fact that $e^r_{\overline{z}, \zeta}$ are the evaluation vectors may be tested against elements $A \in \mathcal{A}_r$ which are given as Toeplitz operators T^r_ϕ on the Hilbert space $H^2(\mathbb{D}, \lambda_r)$ with Γ-invariant symbol ϕ. In this case

$$\tau_{\mathcal{A}_r}(T^r_\phi e^r_{\overline{z}, \zeta}) = \frac{r-1}{\pi} \int_F \phi(\overline{\eta}, \eta) (\sum_\gamma \frac{(1 - |\gamma \eta|^2)^r (1 - \overline{z} \zeta)^r}{(1 - \overline{z} \gamma \eta)^r (1 - \overline{\gamma \eta} \zeta)^r}) \mathrm{d}\lambda_0(\eta)$$

$$= \frac{r-1}{\pi} \int_{\mathbb{D}} \phi(\overline{\eta}, \eta) \frac{(1 - \overline{z} \zeta)^r}{(1 - \overline{z} \gamma \eta)^r (1 - \overline{\gamma \eta} \zeta)^r}) \mathrm{d}\lambda_r(\eta) = < T^r_\phi e^r_z, e^r_\zeta > .$$

This is exactly the symbol of T^r_ϕ evaluated at z, ζ.

Remark 2. *Estimates for the spectral distribution of $e^r_{\overline{z}, \zeta}$, for $z, \zeta \in \mathbb{D}$ may be obtained from*

$$||e^r_{\overline{z}, \zeta}||_2^2 = \tau_{\mathcal{A}_r}(e^r_{\overline{z}, \zeta} e^r_{\overline{\zeta}, z}) = (\frac{r-1}{\pi})^2 \sum_\gamma \frac{(1 - \overline{\zeta} z)^r (1 - \overline{\gamma z} \gamma \zeta)^r}{(1 - \overline{\gamma z} \zeta)^r (1 - \overline{\zeta} \gamma \zeta)^r}, z, \zeta \in \mathbb{D}.$$

We also note the estimates for the higher moments of $e^r_{\overline{z}, \zeta} e^r_{\overline{\zeta}, z}$ although we won't make any use of them. Let $c_r = \frac{r-1}{\pi}$; then

$$\tau_{\mathcal{A}_r}((e^r_{\overline{z}, \zeta} e^r_{\overline{\zeta}, z})^n)$$

$$= (c_r)^{2n} \sum_{\gamma_1,\ldots,\gamma_{2n-1}} \frac{(1-\overline{\gamma_1 z}\gamma_1\zeta)^r(1-\gamma_2 z\overline{\gamma_2\zeta})^r \ldots (1-\overline{\gamma_{2n-1}z}\gamma_{2n-1}1\zeta)^r(1-\overline{z}\zeta)^r}{(1-z\overline{\gamma_1 z})^r(1-\gamma_1\zeta\overline{\gamma_2\zeta})^r(1-\gamma_2 z\overline{\gamma_3 z})^r \ldots (1-\gamma_{2n-1}\zeta\overline{\zeta})^r}.$$

The following estimate holds for the norm in $L^1(\mathcal{A}_r,\tau)$ of $e^r_{\overline{z},\zeta}$:

$$||e^r_{\overline{z},\zeta}||_1 \leq (\frac{r-1}{\pi})^2 (d(z,\zeta))^{-r}, z,\zeta \in \mathbb{D}.$$

Proof. The last estimate may be deduced from the fact that the norm of the evaluation vector $e^r_{\overline{z},\zeta}$ in $L^1(\mathcal{A}_r,\tau)$ should be less then the norm of the corresponding (rank 1) evaluation vector (at z,ζ) on $B(H_r)$. This norm is easily computed to be equal to $(\frac{r-1}{\pi})^2(d(z,\zeta))^{-r}$, for all $z,\zeta \in \mathbb{D}$.

Lemma 3. *Let $A^\Gamma(\overline{\mathbb{D}} \times \mathbb{D})$ be the space of (diagonally) Γ-invariant functions on $\mathbb{D} \times \mathbb{D}$ that are antianalytic in the first variable and analytic in the second variable, with the topology of uniform convergence on compacts subsets of $\mathbb{D} \times \mathbb{D}$. For all ζ, z in $\mathbb{D}$, the following integral is convergent in $A^\Gamma(\overline{\mathbb{D}} \times \mathbb{D})$ and is equal to $e^r_{\overline{z},z}$:*

$$\int_{\mathbb{D}} e^r_{\overline{z},\zeta}(d(z,\zeta))^r d\lambda_0(z) = e^r_{\overline{z},z}.$$

Proof. We will first check the (absolute) uniform convergence on compacts of the integral. We have for all $\eta_1,\eta_2 \in \mathbb{D}$ that:

$$\sup_{\eta_1 \in \mathbb{D}} \int_{\mathbb{D}} \sum_{\gamma} \frac{|1-\overline{\gamma\eta_1}\gamma\eta_2|^r|1-\overline{z}\zeta|^r}{|1-\overline{z}\gamma\eta_2|^r|1-\overline{\gamma\eta_1}\zeta|^r}(d(z,\zeta))^r \mathrm{d}\lambda_0(\zeta)$$

$$\leq \sum_{\gamma} \frac{|1-\overline{\gamma\eta_1}\gamma\eta_2|^r(1-|z|^2)^{r/2}}{|1-\overline{z}\gamma\eta_2|^r} \int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{r/2}}{|1-\overline{\gamma\eta_1}\zeta|^r}\mathrm{d}\lambda_0(\zeta)$$

$$\sum_{\gamma} \frac{|1-\overline{\gamma\eta_1}\gamma\eta_2|^r(1-|z|^2)^{r/2}}{|1-\overline{z}\gamma\eta_2|^r(1-|\gamma\eta_1|^2)^{r/2}}$$

$$= \sum_{\gamma} \frac{|1-\overline{\gamma\eta_1}\gamma\eta_2|^r}{(1-|\gamma\eta_1|^2)^{r/2}(1-|\gamma\eta_2|^2)^{r/2}} \frac{(1-|z|^2)^{r/2}(1-|\gamma\eta_2|^2)^{r/2}}{|1-\overline{z}(\gamma\eta_2)|^r}$$

$$= (d(\eta_1,\eta_2))^{-r} K_r(z,\eta_2).$$

This is quantity uniformly bounded for z,η_1,η_2 in a compact subset of $\mathbb{D}$ ([Be]).

To check the formula in the statement we need to compute

$$\int_{\mathbb{D}} e^r_{\overline{z},\zeta}(\eta_1,\eta_2)(d(z,\zeta))^r \mathrm{d}\lambda_0(\zeta)$$

$$\frac{r-1}{\pi}\int_{\mathbb{D}}\sum_{\gamma}\frac{(1-\overline{\gamma\eta_1}\gamma\eta_2)^r(1-\overline{z}\zeta)^r}{(1-\overline{z}\gamma\eta_2)^r(1-\overline{\gamma\eta_1}\zeta)^r}(d(z,\zeta))^r\mathrm{d}\lambda_0(\zeta)$$

$$=\sum_{\gamma}\frac{(1-\overline{\gamma\eta_1}\gamma\eta_2)^r(1-|z|^2)^{r/2}}{(1-\overline{z}\gamma\eta_2)^r}(\frac{r-1}{\pi})\int_{\mathbb{D}}\frac{(1-\overline{z}\zeta)^{r/2}}{(1-\overline{\gamma\eta_1}\zeta)^r}\frac{1}{(1-z\overline{\zeta})^{r/2}}\mathrm{d}\lambda_{r/2}(\zeta)$$

$$=\sum_{\gamma}\frac{(1-\overline{\gamma\eta_1}\gamma\eta_2)^r(1-|z|^2)^{r/2}}{(1-\overline{z}\gamma\eta_2)^r}\frac{(1-|z|^2)^{r/2}}{(1-\overline{\gamma\eta_1}z)^r}$$

$$=\sum_{\gamma}\frac{(1-\overline{\gamma\eta_1}\gamma\eta_2)^r(1-|z|^2)^r}{(1-\overline{z}\gamma\eta_2)^r(1-\overline{\gamma\eta_1}z)^r}=e^r_{\overline{z},z}(\eta_1,\eta_2).$$

Remark 4. *Note that the formula in the above statement is related to the following equality (valid for any A in $\hat{A}_r$ having the Berezin's symbol $\hat{A}$). We have*

$$(\frac{r-1}{\pi})\int_{\mathbb{D}}A(\overline{z},\zeta)(d(z,\zeta))^r d\lambda_0(\zeta)=A(\overline{z},z),\ z\in\mathbb{D}.$$

Proof. Note that the fact that A is in $\hat{\mathcal{A}}_r$ makes the integral absolutely convergent. We obtain that

$$(\frac{r-1}{\pi})\int_{\mathbb{D}}A(\overline{z},\zeta)(d(z,\zeta))^r\mathrm{d}\lambda_0(\zeta)$$

$$=(\frac{r-1}{\pi})\int_{\mathbb{D}}\frac{A(\overline{z},\zeta)(1-|z|^2)^{r/2}}{(1-\overline{z}\zeta)^{r/2}}\cdot\frac{1}{(1-z\overline{\zeta})^{r/2}}\mathrm{d}\lambda_{r/2}(\zeta)$$

$$\frac{A(\overline{z},z)(1-|z|^2)^{r/2}}{(1-|z|^2)^{r/2}}=A(\overline{z},z).$$

This completes the proof.

For a separable Hilbert space H let $\mathcal{C}_1(H)$ denote the trace class operators on H, with the norm $||\cdot||_1=||\cdot||_{1,\mathcal{C}_1(H)}=||\cdot||_1$. We intend to show that for Γ cocompact subgroup of $PSL(2,\mathbb{R})$ the integral in Lemma 3 is also absolutely convergent in the normic topology of $L^1(\mathcal{A}_r,\tau)$. We will first give a formula to estimate the norm of an element in $L^1(\mathcal{A}_r,\tau)$ in terms of its Berezin symbol.

Lemma 5. *Let Γ be a cocompact subgroup of $PSL(2,\mathbb{R})$. Let A be any element in $L^1(\mathcal{A}_r,\tau)$. Let $\gamma_1,\gamma_2,\ldots$ be an enumeration of G and let F be a fundamental domain for Γ in $\mathbb{D}$. Let G_N be $\cup_{i=1}^{N}\gamma_i F$. Let χ_{G_N} be the characteristic function of G_N viewed as a multiplication operator on $L^2(\mathbb{D},\lambda_r)$.*

Let $||\chi_{G_N}A\chi_{G_N}||_{1,\mathcal{C}_1(L^2(G_N,\lambda_r))}=||\chi_{G_N}A\chi_{G_N}||_1$ be the nuclear norm of the compression of A (viewed as an operator on $L^2(\mathbb{D},\lambda_r)$) to $L^2(G_N,\lambda_r)$.

Then we have the following formula:

$$||A||_{L^1(\mathcal{A}_r,\tau)}=\lim_{N\to\infty}\frac{1}{N}||\chi_{G_N}A\chi_{G_N}||_1.$$

Proof. The normalization $\frac{1}{N}$ comes from the fact that the trace of $\chi_{G_N}A\chi_{G_N}$ acting as a (nuclear) operator on $L^2(G_N,\lambda_r)$ is (by the trace formula in [GHJ]), N times the trace

$$\tau_{\mathcal{A}_r}(A)=\mathrm{tr}_{B(L^2(F,\mathrm{d}\lambda_r))}(\chi_F A\chi_F).$$

We use the identification described in [GHJ] (see also Chapter 3 in [Ra]) of $\pi_r(\Gamma)'$ with $\mathcal{L}(\Gamma)\otimes B(L^2(F,\mathrm{d}\lambda_r))$ acting on $l^2(\Gamma)\otimes L^2(F,\mathrm{d}\lambda_r)$. As proved in [GHJ] the trace of an element x in $L^1(\mathcal{A}_r,\tau)$ is computed by the formula

$$\tau_{\mathcal{A}_r}(x)=\mathrm{tr}_{B(L^2(F,\mathrm{d}\lambda_r))}(\chi_F x\chi_F).$$

In particular

$$||x||_{L^1(\mathcal{A}_r,\tau)}=\mathrm{tr}_{B(L^2(F,\mathrm{d}\lambda_r))}(\chi_F|x|\chi_F).$$

Let P_N be the projection in $B(L^2(\mathbb{D},\lambda_r))$ obtained by multiplication with the characteristic function of χ_{G_N}.

Denote $M=\{\pi_r(\Gamma)\}'$ and let x be any element in $L^1(M,\tau)\cap M$ having support a finite projection in M (like the elements in $L^1(\mathcal{A}_r)$ do, as $\mathcal{A}_r=P_rMP_r$ and P_r is a finite projection in M ([GHJ])). Then $\chi_{G_N}|x|\chi_{G_N}$ is trace class for any N and its trace is, (since $|x|$ commutes with Γ), given by:

$$N[(\mathrm{tr}_{B(L^2(F,\mathrm{d}\lambda_r))}(\chi_F|x|\chi_F)].$$

For any element y in $B(L^2(\mathbb{D},\lambda_r))$ denote the positive and negative part by y_+ and y_- respectively. Then $|P_NxP_N|$, $(P_NxP_N)_+$ are weakly convergent to $|x|$

$x_\pm$ Consequently, for any $k = 1, 2, ...,$ $\chi_{\gamma_k F}(P_N x P_N)_\pm \chi_{\gamma_k F}$ converges weakly to $\chi_{\gamma_k F}(x_\pm)\chi_{\gamma_k F}$.

The trace of $\frac{1}{N}\chi_{G_N}[(P_N x P_N)_\pm]\chi_{G_N}$ is the same as the trace of the element

$$A_N^\pm = \frac{1}{N}\sum_{k=1}^{N} \chi_{(\gamma_k F)}[(P_N x P_N)_\pm]\chi_{(\gamma_k F)}.$$

The trace of $A_N^\pm$ is in turn equal, (by bringing back all this elements under the projection $P_1 = \chi_F$), to the trace $\mathrm{tr}_{B(L^2(F,\mathrm{d}\lambda_r))}(B_N^\pm)$ of the positive element

$$B_N^\pm = \frac{1}{N}\sum_{k=1}^{N} \chi_F\{\pi_r(\gamma_k)^*[\chi_{(\gamma_k F)}[(P_N x P_N)_\pm]\chi_{(\gamma_k F)}](\pi_r(\gamma_k)\}\chi_F.$$

On the other hand since $|x|$ commutes with $\pi_r(\Gamma)$ we have that

$$\frac{1}{N}\sum_{k=1}^{N} \chi_F[\pi_r(\gamma_k)^*(\chi_{(\gamma_k F)}(x_\pm)\chi_{(\gamma_k F)})(\pi_r(\gamma_k)]\chi_F = \chi_F(x_\pm)\chi_F.$$

Since $\chi_{(\gamma_k F)}((P_N x P_N)_\pm)\chi_{(\gamma_k F)}$ converges weakly to $\chi_{(\gamma_k F)}(x_\pm)\chi_{(\gamma_k F)}$ for any k it follows that $B_N^\pm$ converges to $\chi_F(x_\pm)\chi_F$.

Moreover the convergence is dominated; all elements are dominated by a scalar multiple of the positive trace class element $\chi_F P_r \chi_F$ in $B(L^2(F, \lambda_r))$. Hence, by Theorem 2.16 in ([Si]), it follows that

$$\mathrm{tr}_{B(L^2(F,\mathrm{d}\lambda_r))}(\frac{1}{N}\chi_{G_N}((P_N x P_N)_\pm)\chi_{G_N}) = \mathrm{tr}_{B(L^2(F,\mathrm{d}\lambda_r))}(A_N^\pm) = \mathrm{tr}_{B(L^2(F,\mathrm{d}\lambda_r))}(B_N^\pm)$$

converges weakly to $\mathrm{tr}_{B(L^2(F,\mathrm{d}\lambda_r))}(\chi_F(x_\pm)\chi_F)$. This implies that

$$\lim_{N\to\infty} ||\frac{1}{N}\chi_{G_N} x \chi_{G_N})||_{1,\mathcal{C}_1(L^2(G_N,\mathrm{d}\lambda_r))} = \mathrm{tr}_{B(L^2(F,\mathrm{d}\lambda_r))}(\chi_F|x|\chi_F)$$

$$= ||\chi_F x \chi_F||_{1,\mathcal{C}_1(L^2(F,\mathrm{d}\lambda_r))} = ||x||_{1,L^1(\mathcal{A}_r)}.$$

To extend the above result from the class of all x in $L^1(M,\tau) \cap M$ having as support a finite projection in M to the class of all x in $L^1(M,\tau)$ having as support a finite projection in M it is sufficient to observe the following inequality

Note that by the Peierls-Bogoliubov inequality, (also rediscovered by Berezin, see Lemma 8.8 in [Si] and the references therein), we have that, for any positive convex function f with $f(0) = 0$ and x in $M \cap L^1(M, \tau)$, the following inequality holds true

$$\tau_{\mathcal{A}_r}(f(x)) = \mathrm{tr}_{B(L^2(F, \mathrm{d}\lambda_r))}(\chi_F f(x) \chi_F) \geq \mathrm{tr}_{B(L^2(F, \mathrm{d}\lambda_r))}(f(\chi_F x \chi_F)).$$

Hence for any N and modulo a constant depending on r we have

$$\frac{1}{N} ||\chi_{G_N} x \chi_{G_N}||_{1, \mathcal{C}_1(L^2(G_n, \mathrm{d}\lambda_r))} \leq \tau_{\mathcal{A}_r}(|x|).$$

This completes the proof.

We mention the following corollary (without proof) since we are not going to make use of it in this paper. On the other hand it offers a more tractable (for computations) to obtain estimates for elements in the predual of $\mathcal{A}_r$.

Corollary. *With the notations in Lemma 5 we have that for any x in $\mathcal{A}_r = \{\pi_r(\Gamma)\}'$ that*

$$||x||_{L^1(\mathcal{A}_r, \tau)} \leq \text{(const)} \limsup_{N \to \infty} \frac{1}{\sqrt{N}} ||\chi_{G_N} x \chi_{G_N}||_{2, B(L^2(\mathbb{D}, \mathrm{d}\lambda_r))}.$$

contains at least

In the next proposition we will use the above estimate to show that the integral in Lemma 3 is also convergent in $L^1(\mathcal{A}_r, \tau)$.

Lemma 6. *Let Γ be a cocompact, discrete subgroup of $PSL(2, \mathbb{R})$. Let $\mathcal{A}_r$ be the von Neumann algebra of all bounded operators acting on the Hilbert space H_r of the projective representation π_r of $PSL(2, \mathbb{R})$, that commute with $\pi_r(\Gamma)$. Then $\mathcal{A}_r$ is a type II_1 factor ([GHJ]) and $L^2(\mathcal{A}_r, \tau)$ is canonically identified with the Hilbert space of all diagonally $\Gamma-$ invariant functions on $\mathbb{D} \times \mathbb{D}$, antianalytic in the first variable, antianalytic in the second, which are square summable with respect the measure $\frac{\mathrm{d}\lambda_r(z) \mathrm{d}\lambda_r(\zeta)}{(1 - \overline{z}\zeta)^{2r}}$, supported on $\mathbb{D} \times F$. For $z, \zeta \in \mathbb{D}$ let $e^r_{\overline{z}, \zeta} \in L^2(\mathcal{A}_r, \tau)$ be the evaluation vectors at z, ζ.*

Then, for r sufficiently big, the integral

$$\int ||e^r_{\overline{z}, \zeta}||_{L^1(\mathcal{A}_r, \tau)} (d(z, \zeta))^r d\lambda_0(\zeta)$$

is absolutely convergent, uniformly in z in a compact subset of $\mathbb{D}$.

Before that we insert here a disscution on some useful estimates (although they are not of direct use to the proof itself).

We evaluate $||e^r_{\overline{z},\zeta}||_{L^1(\mathcal{A}_r,\tau)}$ for z,ζ in $\mathbb{D}$. Clearly $e^r_{\overline{z},\zeta}$ is the sum of (over Γ) of the operators of rank 1 on $L^2(\mathbb{D},\nu_r)$ given by the formula

$$(1-(\overline{\gamma z})\gamma\zeta)^r < e^r_{\gamma z}, \cdot > e^r_{\gamma\zeta}.$$

It follows that the nuclear norm $||\chi_G e^r_{\overline{z},\zeta}\chi_G||_1$ is bounded by

$$(\frac{r-1}{\pi})\sum_{\gamma\in\Gamma}|1-(\overline{\gamma z})\gamma\zeta|^r||\chi_G e^r_{\gamma z}||_{2,L^2(\mathbb{D},\lambda_r)}||\chi_G e^r_{\gamma\zeta}||_{2,L^2(\mathbb{D},\lambda_r)}$$

$$=(\frac{r-1}{\pi})\sum_{\gamma\in\Gamma}|1-(\overline{\gamma z})\gamma\zeta|^r[\int_G\frac{1}{|1-\overline{\gamma z}\eta|^{2r}}\mathrm{d}\lambda_r(\eta)]^{1/2}[\int_G\frac{1}{|1-\overline{\gamma\zeta}\eta|^{2r}}\mathrm{d}\lambda_r(\eta)]^{1/2}.$$

Hence we have that

$$||\chi_G e^r_{\overline{z},\zeta}\chi_G||_1$$

$$\leq\frac{r-1}{\pi}(d(z,\zeta))^{-r}\sum_{\gamma}[\int_G(d(\gamma z,\eta)^{2r}\mathrm{d}\lambda_r(\eta)]^{1/2}[\int_G(d(\gamma\zeta,\eta)^{2r}\mathrm{d}\lambda_0(\eta)]^{1/2}.$$

Let $\gamma_1,\gamma_2,...$ be an enumeration of Γ and let F be a fundamental domain for Γ in $\mathbb{D}$. Let G_N be $\cup_{i=1}^N\gamma_i F$. Let χ_{G_N} be the characteristic function of G_N viewed as the multiplication operator on $L^2(\mathbb{D},\lambda_r)$. Let $c_r=\frac{r-1}{\pi}$. Consequently,

$$\frac{1}{N}\int_{\mathbb{D}}||\chi_{G_N}e^r_{\overline{z},\zeta}\chi_{G_N}||_{1,B(L^2(F,\lambda_r))}(d(z,\zeta)^r\mathrm{d}\lambda_0(\zeta)$$

$$\leq(c_r)\frac{1}{N}\int_{\mathbb{D}}\sum_{\gamma}[\int_{\cup_{i=1}^N\gamma_i F}(d(\gamma z,\eta)^{2r}\mathrm{d}\lambda_0(\eta)]^{1/2}[\int_{\cup_{i=1}^N\gamma_i F}(d(\gamma\zeta,\eta)^{2r}\mathrm{d}\lambda_0(\eta)]^{1/2}\mathrm{d}\lambda_0(\zeta)$$

$$=c_r\int_{\mathbb{D}}\sum_{\gamma}[\frac{1}{N}\sum_{i=1}^N\int_{\gamma_i F}(d(\gamma z,\eta)^{2r}\mathrm{d}\lambda_0(\eta)]^{1/2}[\frac{1}{N}\sum_{i=1}^N\int_{\gamma_i F}(d(\gamma\zeta,\eta)^{2r}\mathrm{d}\lambda_0(\eta)]^{1/2}\mathrm{d}\lambda_0(\zeta).$$

We denote the last sums by $\phi_N(z)$. Assume that 0 belongs to F. It is then easy to see, by the arguments in ([Le]), that for all $\zeta\in\mathbb{D}$, $\sigma\in\Gamma$, $\int_F(d(\zeta,\sigma\eta)^{2r}\mathrm{d}\lambda_0(\eta)=$ $\int_F(d(\sigma\zeta,\eta)^{2r}\mathrm{d}\lambda_0(\eta)$ is comparable (uniformly in $\zeta\in\mathbb{D}$ and $\sigma\in\Gamma$) with $d(\zeta,\sigma 0)$

Hence (modulo a constant depending on Γ and r), $\phi_N(z)$ is dominated by

$$\sum_{\gamma}[\frac{1}{N}\sum_{i=1}^{N} d(\gamma z, \gamma_i 0)^{2r}]^{1/2}\int_{\mathbb{D}}[\frac{1}{N}\sum_{i=1}^{N} d(\gamma\zeta, \gamma_i 0)^{2r}]^{1/2}\mathrm{d}\lambda_0(\zeta).$$

In turn this quantity (because $\mathrm{d}\lambda_0$ is an invariant measure) is equal to

$$\sum_{\gamma}[\frac{1}{N}\sum_{i=1}^{N} d(\gamma z, \gamma_i 0)^{2r}]^{1/2}\int_{\mathbb{D}}[\frac{1}{N}\sum_{i=1}^{N}(d(\zeta, \gamma_i 0))^{r}]^{1/2}\mathrm{d}\lambda_0(\zeta)$$

$$=\int_{\mathbb{D}}[\frac{1}{N}\sum_{i=1}^{N}(d(\zeta, \gamma_i 0))^{r}]^{1/2}\mathrm{d}\lambda_0(\zeta)\sum_{\gamma}[\frac{1}{N}\sum_{i=1}^{N} d(\gamma z, \gamma_i 0)^{2r}]^{1/2}.$$

The integral $\int_{\mathbb{D}}[\frac{1}{N}\sum_{i=1}^{N}(d(\zeta, \gamma_i 0))^{r}]^{1/2}\mathrm{d}\lambda_0(\zeta)$ is

$$\sum_{\sigma\in\Gamma}\int_{\sigma F}[\frac{1}{N}\sum_{i=1}^{N} d(\zeta, \gamma_i 0)^{r}]^{1/2}\mathrm{d}\lambda_0(\zeta)$$

$$=\sum_{\sigma\in\Gamma}\int_{F}[\frac{1}{N}\sum_{i=1}^{N} d(\sigma\zeta, \gamma_i 0)^{r}]^{1/2}\mathrm{d}\lambda_0(\zeta).$$

Modulo a constant, the last integral is comparable (uniformly in $\gamma_1, \gamma_2, ...$ and N) to

$$\sum_{\sigma\in\Gamma}(\frac{1}{N}\sum_{i=1}^{N} d(\sigma 0, \gamma_i 0)^{r})^{1/2}.$$

Hence, we get that (modulo a constant depending only on Γ and r)

$$\frac{1}{N}\int_{\mathbb{D}}||\chi_{G_N} e^{r}_{\overline{z},\zeta}\chi_{G_N}||_{1,\mathcal{C}_1(L^2(\mathbb{D},\lambda_r))}(d(z.\zeta))^{r}\mathrm{d}\lambda_0(\zeta)$$

$$\leq \mathrm{const}_{r,\Gamma}\{\sum_{\gamma}[\frac{1}{N}\sum_{i=1}^{N} d(\gamma 0, \gamma_i 0)^{2r}]^{1/2}\}^{2}.$$

Let

$$y_N=\sum_{\gamma}[\frac{1}{N}\sum_{i=1}^{N} d(\gamma 0, \gamma_i 0)^{2r}]^{1/2},\ N\in\mathbb{N}.$$

To be able to take N to limit one should have that the above sums are uniformly bounded in N. Note that if the summand for Γ wouldn't be raised to the power $1/2$ then this would have been (by Γ invariance)

$$\sum_{\gamma}(\frac{1}{N}\sum_{i=1}^{N} d(\gamma 0, \gamma_i 0)^{2r}) = \sum_{\gamma}(\frac{1}{N}\sum_{i=1}^{N} d(\gamma_i\gamma 0, 0)^{2r})$$

$$= N\sum_{\gamma}(\frac{1}{N}\sum_{i=1}^{N} d(\gamma 0, 0)^{2r}) = \sum_{\gamma}(d(\gamma 0, 0))^{2r})$$

which is finite by the arguments in [Be] as soon as k is bigger then 1.

We use the notations and the methods in the survey article by Lehner. Let $n(r,0)$ be the numbers of points in the orbit of $\Gamma 0$ contained in the euclidian disk of radius r in $\mathbb{D}$. By Tsuji estimates ([Ts]) and by the asymptotic formula of Huber ([Hu]) $n(r,0)$ is asymptotically $\frac{1}{2g-1}\frac{1}{1-r}$, where g is the genus of the compact Riemann surface $\mathbb{D}/\Gamma$. Also the distribution of the orbit $\Gamma 0$ is uniform with respect to arc measure ([EM]). I am very indebted to C. T. McMullen for giving me this information (and many other informations which in the end weren't directly related to this paper).

Neglecting the cardinality of the stabilizer of 0, which is finite, and by using the arguments in [Le] (in the argument of Theorem 2.2.5 loc. cit) we get

Remark. *Let $\gamma_1, \gamma_2, ...$ is an enumeration of Γ so that $d(0,\gamma_i 0)^{-1}$ is increasing. Let $N = n(s_0, 0)$ and let y_N be defined as in (2) by*

$$y_N = \sum_{\gamma}[\frac{1}{N}\sum_{i=1}^{N} d(\gamma 0, \gamma_i 0)^{2r}]^{1/2}, \ N \in \mathbb{N}.$$

Then, modulo a constant, we have that y_N is asymptotically equal to

$$\int_0^1\int_0^{2\pi}[n(s_0,0)^{-1}\int_0^{s_0}\int_0^{2\pi}(\frac{(1-r)^k(1-s)^k}{(1-rs\exp i(\phi-\theta))^{2k}}d\phi d(n(s,0))]^{1/2}d\theta\, d(n(r,0))$$

$$= \int_0^1[(1-s_0)\int_0^{s_0}\frac{(1-r)^k(1-s)^k}{(1-rs)^{2k-1}}d(n(s,0))]^{1/2}d(n(r,0)).$$

It is easy to see that, by using the fact that the distribution $\mathrm{d}(n(r,0))$ is asymptotically const $\frac{1}{\ldots}$ that if we add factor $1/\sqrt{N}$ in the sum in (2) or if we add a

factor $\sqrt{(1-s_0)}$ in front of the integral representing the sum then we would be able to find a finite upper bound which is valid for all N.

This is because the integral representing the sums is dominated by terms of the form $(1-s_0)^{-1/2}$ (see the computations bellow). To get rid of this (unfortunate) power of N in our estimate we have thus to use a better estimate for the integral in our statement.

Proof (of Proposition 6). We neglect the cardinality of any stabilizer (because these are finite ([Le])). We will use the distribution function $n(r,\theta)$ counting the number of points from the orbit $\Gamma 0$ which are in a sector of radius r and angle θ from the origin. Let

$$\gamma = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix};\ \gamma_1 = \begin{pmatrix} a_1 & b_1 \\ \overline{b_1} & \overline{a_1} \end{pmatrix},$$

$$\gamma_1^{-1} = \begin{pmatrix} \overline{a_1} & -b_1 \\ -\overline{b_1} & a_1 \end{pmatrix} \quad \gamma\gamma_1 = \begin{pmatrix} aa_1 + b\overline{b_1} & ab_1 + b\overline{a_1} \\ \overline{b}a_1 + \overline{a}\overline{b_1} & \overline{b}b_1 + \overline{a}\overline{a_1} \end{pmatrix}.$$

Let $\gamma 0 = r\exp(i\theta) = \frac{b}{\overline{a}}$, $\gamma_1 0 = s\exp(i\phi) = \frac{b_1}{\overline{a_1}}$. Hence

$$(\gamma\gamma_1)0 = \frac{ab_1 + b\overline{a_1}}{\overline{b}b_1 + \overline{a}\overline{a_1}} = \frac{\frac{b_1}{\overline{a_1}} + \frac{b}{\overline{a}}\frac{\overline{a}}{a}}{\frac{b_1}{\overline{a_1}}\frac{\overline{b}}{a}\frac{a}{\overline{a}} + 1} \cdot \frac{a\overline{a_1}}{\overline{a_1}\overline{a}} = \frac{s\mathrm{e}^{i(\phi+\alpha(\gamma))} + r\mathrm{e}^{i\theta}}{1 + rs\mathrm{e}^{i(\phi+\alpha(\gamma)-\theta)}}.$$

We will use the notation:

$$\exp(i\alpha(\gamma)) = \exp(i\alpha(\gamma 0)) = \frac{a}{\overline{a}},\ \text{if } \gamma = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \in \Gamma.$$

Note that

$$\gamma^{-1}0 = \begin{pmatrix} \overline{a} & -b \\ -\overline{b} & a \end{pmatrix} 0 = \frac{b}{a} = \frac{b}{\overline{a}}\frac{\overline{a}}{a} = (\gamma 0)\mathrm{e}^{-i\alpha(\gamma)} = r\mathrm{e}^{i(\theta-\alpha(\gamma))}.$$

Then

$$(\gamma\gamma_1)0 = \frac{r\exp(i(\theta-\alpha(\gamma))) + s\exp(i\phi)}{1 + rs\exp(i(\phi+\alpha(\gamma)-\theta))}\exp(i\alpha(\gamma)).$$

Also note that in this case

$$= \frac{(1-|\gamma_1 0|^2)(1-|\gamma^{-1}0|^2)}{|1-\overline{\gamma_1 0}(\gamma^{-1}0)|^2} = \frac{(1-r^2)(1-s^2)}{|1+rs\mathrm{e}^{i(-\phi+\theta-\alpha(\gamma))}|^2}.$$

In what follows we will use the notations

$$\aleph(\gamma 0) = \aleph(r\mathrm{e}^{i\theta}) = \mathrm{e}^{i(\pi+\alpha(\gamma))}.$$

Hence, with $Z = r\mathrm{e}^{i\theta}$, $\zeta = \gamma_1 0$, we have that

$$\gamma\gamma_1 0 = \frac{Z - \aleph(Z)\zeta}{1-\aleph(Z)\overline{Z}\zeta}.$$

Note that if denote $\gamma\gamma_1 0 = f_{\gamma_1}(\gamma 0)$, then we must have that for all γ in Γ that

$$\sigma(f_{\gamma_1}(\gamma 0)) = f_{\gamma_1}(\sigma\gamma 0),$$

i. e. that

$$\sigma[\frac{Z-\aleph(Z)\zeta}{1-\overline{Z}\aleph(Z)\zeta}] = \frac{\sigma(Z)-\aleph(\sigma(Z))\zeta}{1-\overline{Z}\aleph(\sigma(Z))\zeta}, \tag{1}$$

for all Z in the orbit $\Gamma 0$ and all σ in Γ.

For functions f on $\mathbb{D}$, we have that

$$\sum_{\gamma\in G}(f(\gamma 0)) = \sum_{\gamma\in\Gamma}(f(\gamma^{-1}0)).$$

By using the density distribution $\mathrm{d}n(r,\theta)$ counting the points the orbit $\Gamma 0$ in a sector in $\mathbb{D}$ of radius r and angle θ (see [Le],[EM]) we get that

$$\int_{\mathbb{D}}(f(Z\aleph(Z))\mathrm{d}\lambda_0(Z) - \int_{\mathbb{D}} f(Z)\mathrm{d}\lambda_0(Z)$$

tends to zero when the support of f is close to the boundary of $\mathbb{D}$ (modulo terms of lower order with respect to the distance to the boundary).

Let $\gamma,\gamma_1 \in \Gamma$ and denote $Z = \gamma 0, \zeta = \gamma_1 0$. Then

$$1-\gamma\gamma_1 0(\overline{\gamma 0}) = 1 - \overline{Z}\frac{Z-\aleph(Z)\zeta}{1-\aleph(Z)\overline{Z}\zeta} = \frac{(1-|Z|^2)}{1-\aleph(Z)\overline{Z}\zeta}, \tag{2}$$

and

$$|1-\gamma\gamma_1 0(\overline{\gamma 0})| = \frac{(1-|Z|^2)}{|1-\aleph(Z)\overline{Z}\zeta|}. \tag{3}$$

Also we have for all η_1 in $\mathbb{D}$ that

$$(1-(\overline{\eta_1})\gamma\zeta)^r = (1-\overline{\eta_1}(\gamma\gamma_1 0))^r = [1-\overline{\eta_1}\frac{Z-\aleph(Z)\zeta}{1-\overline{Z}\aleph(Z)\zeta}]^r$$

$$= \frac{(1-\overline{Z}\aleph(Z)\zeta-\overline{\eta_1}(Z-\aleph(Z)\zeta))^r}{(1-\overline{Z}\aleph(Z)\zeta)6r}. \tag{4}$$

With this notations ($Z=\gamma 0$, $\zeta=\gamma_1 0$) we have that

$$(1-|\gamma\gamma_1 0|^2) = \frac{(1-|Z|^2)(1-|\zeta|)^2}{|1-\overline{Z}\aleph(Z)\zeta|^2}. \tag{5}$$

Recall that for arbitrary z,ζ in $\mathbb{D}$ we have that

$$e^r_{\overline{z},\zeta}(\overline{\eta_1},\eta_2) = |d(\overline{z},\zeta)|^{-r}\sum_{\gamma\in\Gamma}\frac{(1-\overline{\gamma z}(\gamma\zeta))^r}{|1-\overline{\gamma z}(\gamma\zeta)|^r}\frac{(1-\overline{\eta_1}\eta_2)^r(1-|\gamma z|^2)^{r/2}(1-|\gamma\zeta|^2)^{r/2}}{(1-\overline{\gamma z}\eta_2)^r(1-\overline{\eta_1}\gamma\zeta)^r},$$

for all $\eta_i\in\mathbb{D}$. Note that the factor $|d(\overline{z},\zeta)|^{-r}$ which we get in front of the formula for $e^r_{\overline{z},\zeta}$ will be canceled by the corresponding factor $|d(\overline{z},\zeta)|^r$ in the formula for $||e^r_{\overline{z},\zeta}||_{\lambda,r}$.

The formula which we therefore get for $e^r_{\overline{z},\zeta}$ for $z=0$, $\zeta=\gamma_1 0$ is

$$e^r_{0,\zeta}(\eta_1,\eta_2) = (|d(0,\zeta)|^{-r})\sum_{\gamma\in\Gamma}\frac{(1-\overline{\gamma 0}(\gamma\gamma_1 0))^r}{|1-\overline{\gamma 0}\gamma\gamma_1 0|^r}\frac{(1-\overline{\eta_1}\eta_2)^r(1-|\gamma 0|^2)^{r/2}(1-|\gamma\gamma_1 0|^2)^{r/2}}{(1-\overline{\gamma 0}\eta_2)^r(1-\overline{\eta_1}(\gamma\gamma_1 0))^r},$$

for all $\eta_i\in\mathbb{D}$. We now use the method in Lehner ([Le]) to express the sum after γ as an integral. We use the formulae (2), (3), (4) (5) above.

By using the notation $\zeta=\gamma_1 0$ and $Z=re^{i\theta}$ for the integration variable (r,θ are the variables for the density function $\mathrm{d}n(r,\theta)$ which counts the number of points in the orbit of $\Gamma 0$ in a sector of radius r and angle θ in $\mathbb{D}$), we get

$$e^r_{0,\gamma_1 0}(\eta_1,\eta_2) = e^r_{0,\zeta}(\eta_1,\eta_2)$$

$$= (|d(0,\zeta)|^{-r})\int\frac{(1-\overline{\eta_1}\eta_2)^r(1-|Z|^r)(1-|\zeta|^2)^{r/2}}{(1-\overline{Z}\eta_2)^r(1-\zeta\aleph(Z)\overline{Z}-\overline{\eta_1}(Z-\aleph(Z)\zeta))^r}\mathrm{d}n(r,\theta).$$

For our estimates we may replace the measure $\mathrm{d}n(r,\theta)$ by $r\mathrm{d}n(r,\theta)$ which means that a sufficiently good approximation for our purposes for $e^r_{\overline{z},\zeta}$ will be

(6)

$$G_\zeta(\overline{\eta_1},\eta_2) = (|d(0,\zeta)|^{-r})\int_{\mathbb{D}} \frac{(1-\overline{\eta_1}\eta_2)^r(1-|\zeta|^2)^{r/2}}{(1-\overline{Z}\eta_2)^r(1-\zeta\aleph(Z)\overline{Z}-\overline{\eta_1}(Z-\aleph(Z)\zeta)^r}\mathrm{d}\lambda_r(Z).$$

The resulting formula, has by the invariance property in (1) has the property that

$$G(\overline{\gamma\eta_1},\gamma\eta_2) = G(\overline{\eta_1},\eta_2).$$

If we further particularize to $\eta_1=\sigma_1 0, \eta_2=\sigma_2 0$ then we get

$$a_{\sigma_1^{-1}\sigma_2} = G(\overline{\sigma_1 0},\sigma_2 0) = G(0,\sigma_1^{-1}\sigma_2 0)$$

$$= (|d(0,\zeta)|^{-r})\int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{r/2}}{(1-\overline{Z}(\sigma_1^{-1}\sigma_2 0))^r(1-\zeta\aleph(Z)\overline{Z})^r}\mathrm{d}\lambda_r(Z). \tag{7}$$

We use (7) to find estimates on $||e^r_{0,\zeta}||_1$ when ζ tends to the boundary of $\mathbb{D}$. By Proposition 5 we may estimate the norm $||\cdot||_1$ for an element whose Berezin ($\Gamma-$invariant) kernel is $k=k(\overline{\eta_1},\eta_2)$ by the norm of the operator on $l^2(\Gamma)$ given by the matrix

$$A=(A_{\sigma_1,\sigma_2}); A_{\sigma_1,\sigma_2} = |d(\sigma_1 0,\sigma_2 0)|^r k(\overline{\sigma_1 0},\sigma_2 0).$$

Let δ_γ be the left convolutor by γ on $l^2(\Gamma)$. Hence to estimate $(|d(0,\zeta)|^r)||e^r_{0,\zeta}||_1$ we could use, by (7), the norm 1 of the following element in the predual of $\mathcal{L}(\Gamma)$:

$$\sum_{\sigma\in\Gamma}(1-|\sigma 0|^2)^{r/2}\{\int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{r/2}}{(1-\overline{Z}\sigma_1^{-1}\sigma_2 0)^r(1-\zeta\aleph(Z)\overline{Z})^r}\mathrm{d}\lambda_r(Z)\}\delta_\gamma.$$

We observe that if replace $\aleph(Z)$ by 1 then in the above integral the power of ζ in the denominator disappears in the integral after Z. Thus if we replace $\aleph(Z)$ by 1 we get an a convergent integral for $\int_{\mathbb{D}}(|d(0,\zeta)|^r)||e^r_{0,\zeta}||_1\mathrm{d}\lambda_r(\zeta)$.

We have proved above that integrals involving $\aleph(Z)$ behave, for functions whose support tends to the boundary of $\mathbb{D}$, like $\aleph(Z)$ tends to 1. The remainder, by making this approximation just brings an additional order of zero in the following integrals which estimate the integral over $\mathbb{D}$ of the first Sobolev norm of $(|d(0,\zeta)|^r)e^r_{\overline{z},\zeta}$. This additional power of zero will allow to estimate the norm $||\cdot||_1$ (as, by ([BS], [St]), the Sobolev $(1+\epsilon)$ norm is dominating the norm $||\cdot||_1$. The proof of Proposition 7 is complete, once we go through the following Lemma.

Proposition 7. *Let $\sigma_1, \sigma_2, ...$ an enumeration of Γ and let F be a fundamental domain for Γ in $\mathbb{D}$ and let $G_N = \cup_{i=1}^N \sigma_i F$. Then the following integrals are (absolutely) convergent, uniformly in $N \in \mathbb{N}$ and z in $\mathbb{D}$.*

$$(3) \qquad \sup_{N\in\mathbb{N}} \int_{\mathbb{D}} \frac{1}{N} ||\aleph_{G_N} e^r_{z,\zeta} \aleph_{G_N}||_{2,\mathcal{C}_2(L^2(G_N,\mathrm{d}\lambda_r))} (d(z,\zeta))^r \, d\lambda_0(\zeta) < \infty.$$

Moreover the above expression is bounded, (modulo constants that only depend on Γ and r), by the following (finite) quantity:

$$\sup_{N\in\mathbb{N}} \sum_{\gamma_1} [\frac{1}{N^2} \sum_{i,j=1}^N |\sum_{\gamma} |d(\gamma 0, \sigma_i 0)|^r |d(\sigma_j 0, \gamma\gamma_1 0)|^r|^2]^{1/2} < \infty.$$

Note that the kernel representing the operator $e^r_{\overline{z},\zeta}$ on $L^2(G_N, d\lambda_r)$ is $(1-\overline{\eta_1}\eta_2)^{-r} e^r_{\overline{z},\zeta}(\eta_1,\eta_2)$. Let $f^{r,N}_{\overline{z},\zeta}$ be the partial derivative, after η_1, of the kernel representing $e^r_{\overline{z},\zeta}$ on $L^2(G_N, d\lambda_r)$. Thus

$$f^{r,N}_{\overline{z},\zeta}(\overline{\eta_1},\eta_2) = \aleph_{G_N}(\eta_1)\aleph_{G_N}(\eta_2) \frac{d}{d\eta_1} (1-\overline{\eta_1}\eta_2)^{-r} e^r_{\overline{z},\zeta}(\overline{\eta_1},\eta_2), \eta_1,\eta_2 \in \mathbb{D}.$$

Then the following integrals are absolutely convergent, uniformly in N (and $z \in \mathbb{D}$):

$$\sup_{N\in\mathbb{N}} \int_{\mathbb{D}} \frac{1}{N} ||f^{r,N}_{z,\zeta}||_{(L^2(G_N,\mathrm{d}\lambda_r))^2} (d(z,\zeta))^r \, d\lambda_0(\zeta) < \infty.$$

Similarly, this is bounded, (modulo constants that only depend on r and Γ), uniformly in $N \in \mathbb{N}$, by

$$\sup_{N\in\mathbb{N}} \sum_{\gamma_1} [\frac{1}{N^2} \sum_{i,j=1}^N |\sum_{\gamma} \frac{1}{(1-\overline{(\sigma_i 0)}\gamma 0)} \frac{|1-\overline{\gamma 0}(\gamma\gamma_1 0)|}{(1-\overline{\gamma 0}(\gamma\gamma_1 0))} (\tilde{d}(\overline{\sigma_i 0},\gamma 0))^r (\tilde{d}(\overline{\gamma\gamma_1 0},\sigma_j 0))^r|^2]^{1/2} < \infty.$$

Proof. We have

$$\frac{1}{N^2} ||\aleph_{G_N} e^r_{z,\zeta} \aleph_{G_N}||^2_{2,\mathcal{C}_2(L^2(G_N,\mathrm{d}\lambda_r))}$$

$$= \frac{1}{N^2} \int_{G_N} \int_{G_N} |\sum \frac{(1-\overline{\eta_1}\eta_2)^r (1-\overline{\gamma z}(\gamma\zeta))^r}{(1-\overline{\gamma z}\eta_2)^r (1-\overline{\eta_1}(\gamma\zeta))^r} \cdot \frac{1}{(1-\overline{\eta_1}\eta_2)^r} |^2 \mathrm{d}\lambda_r(\eta_1,\eta_2)$$

$$= \frac{1}{N^2}\sum_{i,j=1}^{N}\int_{\sigma_i F}\int_{\sigma_j F}|\sum_{\gamma}|d(\gamma z,\gamma\zeta)|^{-r}|d(\gamma z,\eta_2)|^r|d(\eta_1,\gamma\zeta)|^r|^2\mathrm{d}\lambda_0(\eta_1,\eta_2).$$

If, as we did above, we replace any integral over the fundamental domain F with the value of the function to be integrated at 0 we get that, modulo a constant,

$$\frac{1}{N^2}||\aleph_{G_N} e^r_{z,\zeta}\aleph_{G_N}||^2_{2,\mathcal{C}_2(L^2(G_N,\mathrm{d}\lambda_r))}$$

is bounded by the following

$$(d(z,\zeta))^{-2r}\frac{1}{N^2}\sum_{i,j=1}^{N}|\sum_{\gamma}d(\gamma z,\sigma_i 0)^r d(\sigma_j 0,\gamma\zeta)^r|^2.$$

The integral in the statement may, by the same arguments as in the comments after the statement of Proposition 6, be compared, (uniformly in z in a compact set), by a discrete sum over Γ. Hence the integral in our statement is bounded, (modulo a constant which depends only on Γ and which also depends (continuously) on r), by the following sum:

$$\sum_{\gamma_1}[\frac{1}{N^2}\sum_{i,j=1}^{N}|\sum_{\gamma}|d(\gamma 0,\sigma_i 0)|^r|d(\sigma_j 0,\gamma\gamma_1 0)|^r|^2]^{1/2}.$$

Here, again, we have replaced the integrals over F by the value of the integrand at 0. The integral in which we replace $(1-\overline{\eta_1}\eta_2)^{-r}e^r_{\overline{z},\zeta}(\eta_1,\eta_2)$ by its partial derivative with respect to η_1:

$$\frac{\mathrm{d}}{\mathrm{d}\eta_1}(1-\overline{\eta_1}\eta_2)^{-r}e^r_{\overline{z},\zeta}(\overline{\eta_1},\eta_2),$$

will be bounded by a similar sum, which carries the additional factor:

$$\frac{1}{(1-\overline{\sigma_i 0}(\gamma 0))}.$$

If we use the next lemma, this completes the proof of Proposition 7.

Lemma. *Let Γ be a cocompact subgroup of $SU(1,1)$. Let $d(z,\zeta)$ be the square root of the hyperbolic distance between two points z,ζ in $\mathbb{D}$, i.e.*

$$d(z,\zeta) = (1-|z|^2)^{1/2}(1-|\zeta|^2)^{1/2}|1-\overline{z}\zeta|^{-1}$$

Let $\sigma_1, \sigma_2, \ldots$ be an enumeration of Γ. Let $\tilde{d}(\overline{z}, w) = \frac{(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}}{(1-\overline{z}w)}$, for all $z, w \in \mathbb{D}$. The following sums are absolutely convergent, uniformly with $N \in \mathbb{N}$:

$$\sup_{N\in\mathbb{N}} \sum_{\gamma_1} [\frac{1}{N^2} \sum_{i,j=1}^{N} |\sum_{\gamma} |d(\gamma 0, \sigma_i 0)|^r |d(\sigma_j 0, \gamma\gamma_1 0)|^r|^2]^{1/2} < \infty \tag{8}$$

The same holds true (uniformly in $N \in \mathbb{N}$) for the following sums

(9)

$$\sup_{N\in\mathbb{N}} \sum_{\gamma_1} [\frac{1}{N^2} \sum_{i,j=1}^{N} |\sum_{\gamma} \frac{1}{(1-\overline{(\sigma_i 0)}\gamma 0)} \frac{|1-\overline{\gamma 0}(\gamma\gamma_1 0)|}{(1-\overline{\gamma 0}(\gamma\gamma_1 0))} (\tilde{d}(\overline{\sigma_i 0}, \gamma 0))^r (\tilde{d}(\overline{\gamma\gamma_1 0}, \sigma_j 0))^r|^2]^{1/2} < \infty.$$

We replace the sums $\sum_\gamma$ and $\sum_{\gamma_1}$ in (8) and (9) by the integral over $\mathbb{D}$ with respect to the densities $\mathrm{d}(n(a_2, \theta_2))$ and respectively $\mathrm{d}(n(a_1, \theta_1))$. Also we replace the sums $\sum_{\sigma_1}$, $\sum_{\sigma_2}$ by $\mathrm{d}(n(t_1, \phi_1))$ and $\mathrm{d}(n(t_2, \phi_2))$ respectively. We also let $N = \frac{1}{1-s}$. We use the notations from the proof of Proposition 6. By using the method in ([Le]) we get that the supreme after N of the sum (8) in the statement is finite if and only if the supreme over s of the following integrals is finite (for the convenience of the notation we will replace r by $2r$):

$$\int_0^1 \int_0^{2\pi} [(1-s)^2 \int_0^s \int_0^{2\pi} \int_0^s \int_0^{2\pi} | \int_0^1 \int_0^{2\pi} \frac{(1-a_2)^r (1-t_1)^r}{|1-t_1 a_2 \exp(i(\phi_1-\theta_2))|^{2r}} \cdot$$

$$\frac{(1-t_2)^r (1-|\gamma\gamma_1 0|^2)^r}{|1-t_2(\gamma\gamma_1 0)|^{2r}} \frac{\mathrm{d}\theta_2 \mathrm{d}a_2}{(1-a_2)^2}|^2 \frac{\mathrm{d}\phi_1 \mathrm{d}t_1}{(1-t_1)^2} \frac{\mathrm{d}\phi_2 \mathrm{d}t_2}{(1-t_2)^2}]^{1/2} \frac{\mathrm{d}\theta_1 \mathrm{d}a_1}{(1-a_1)^2}.$$

$$= \int_0^1 \int_0^{2\pi} [(1-s)^2 \int_0^s \int_0^{2\pi} \int_0^s \int_0^{2\pi} | \int_0^1 \int_0^{2\pi} \frac{(1-a_2)^r (1-t_1)^r}{|1-t_1 a_2 \exp(i(\phi_1-\theta_2))|^{2r}} \cdot$$

$$\frac{(1-t_2)^r (1-a_1)^r (1-a_2)^r}{|1+a_1 a_2 \mathrm{e}^{i(\alpha(\gamma)-\theta_2+\theta_1)} - t_2 \mathrm{e}^{-i\phi_2}(a_2 \mathrm{e}^{i\theta_2} + a_1 \mathrm{e}^{i(\alpha(\gamma)+\theta_1)})|^{2r}} \cdot$$

$$\cdot \frac{\mathrm{d}\theta_2 \mathrm{d}a_2}{(1-a_2)^2}|^2 \frac{\mathrm{d}\phi_1 \mathrm{d}t_1}{(1-t_1)^2} \frac{\mathrm{d}\phi_2 \mathrm{d}t_2}{(1-t_2)^2}]^{1/2} \frac{\mathrm{d}\theta_1 \mathrm{d}a_1}{(1-a_1)^2}.$$

The integral in which we replace $(1-\overline{\eta_1}\eta_2)^{-r} e^r_{\overline{z},\zeta}(\overline{\eta_1}, \eta_2)$ by its partial derivative

$$\frac{\mathrm{d}}{\mathrm{d}} (1-\overline{\eta_1}\eta_2)^{-r} e^r_{\overline{z},\zeta}(\overline{\eta_1}, \eta_2)$$

with respect η_1, carries similar terms. The term which comes from the derivation will add a factor, in the summands, of the form $\frac{1}{(1-\overline{\sigma_i 0}(\gamma 0))}$, corresponding to the differentiation of $(1-\overline{\eta_1}\zeta)^{-r}$. In the above integral this will bring an additional factor of the form

$$\frac{1}{(1-t_1 a_2 e^{i(\phi_1-\theta_2)})}.$$

Our arguments thus allow to estimate the sum in (9) by

$$\int_0^1\int_0^{2\pi}[(1-s)^2\int_0^s\int_0^{2\pi}\int_0^s\int_0^{2\pi}|\int_0^1\int_0^{2\pi}\frac{(1-a_1)^r(1-t_1)^r(1-t_2)^r}{(1-t_1a_2e^{i(\phi_1-\theta_2)})^{2r+1}}\cdot$$

$$\cdot\frac{(1-a_2)^{2r-2}}{\{1-a_1a_2\mathrm{e}^{i(\alpha(\gamma)-\theta_2+\theta_1)}-t_2\mathrm{e}^{-i\phi_2}(a_2\mathrm{e}^{i\theta_2}+a_1\mathrm{e}^{i(\alpha(\gamma)+\theta_1)})\}^{2r}}\cdot$$

$$\cdot\mathrm{d}\theta_2\mathrm{d}a_2|^2\frac{\mathrm{d}\phi_1\mathrm{d}t_1}{(1-t_1)^2}\frac{\mathrm{d}\phi_2\mathrm{d}t_2}{(1-t_2)^2}]^{1/2}\frac{\mathrm{d}\theta_1\mathrm{d}a_1}{(1-a_1)^2}.$$

By Stinespring's estimates ([St]) for the trace class norms (applied to the terms of the form $\aleph_{G_N}e^r_{\overline{z},\zeta}\aleph_{G_N}$), the integrals in (8) and (9) above (the one for $e^r_{\overline{z},\zeta}$ and the one corresponding to its partial derivative), will bound the integral in the statement of Proposition 7.

This holds because of Theorem 2 in ([St]) which shows that we may choose the same constant in the estimates bounding the nuclear norms for operators on the Hilbert spaces $L^2(G_N,\mathrm{d}\lambda_r)$ by the L^2 norm on G_N of the first derivative of the kernel representing the operator (plus the Hilbert-Schmidt norm). The Stinespring's theorem applies here because $\mathrm{d}\lambda_r$ is a finite measure on the compact space $\mathbb{D}$.

Moreover, the renormalization we have to perform on $e^r_{\overline{z},\zeta}$ (i.e to divide $e^r_{\overline{z},\zeta}(\overline{\eta_1},\eta_2)$ by $(1-\overline{\eta_1}\eta_2)^r$) comes from the fact that if the Berezin kernel of an operator is $k(\overline{\eta_1},\eta_2)$ then this operator is in fact represented by the kernel $\frac{k(\overline{\eta_1},\eta_2)}{(1-\overline{\eta_1}\eta_2)^r}$.

We will show bellow that for r sufficiently big, the integrals are uniformly bounded in $s\in(0,1)$. The two integrals bounding (8) and (9) both carry terms of the form

$$\frac{\text{product of factors } (1-(f(a_1,a_2,t_1,t_2,s))^{a_f}}{\text{[illegible]}},$$

where the functions f, g tend to 1 as the parameters tend to 1. Moreover the factors on the top of the fraction behave like products of terms of the form

$$(1-a_i)^{\alpha_i}(1-t_i)^{\beta_i}(1-s)^{\delta}$$

while the terms on the bottom of the fraction behave like that after taking out the angle measures ϕ_i, θ_i, $\alpha(\gamma)$.

We use the following convention to denote an integral of the above form (or an homegenuous sum of such integrals) by $[A-B]$ where A is the total degree of the factors on the top, i.e A is the sum

$$A = \sum_{i=1,2} \alpha_i + \sum_{i=1,2} \beta_i + \delta,$$

and similarly for B.

At each of the partial stages in the integration process for the integrals bounding (8) and (9) we will get similar integrals, with one variable from the set (a_1, t_1, t_2, a_2) (or an angle variable) missing.

The effect of the integration with respect to $\frac{\mathrm{d}a_i}{(1-a_i)^2}$ and $\frac{\mathrm{d}t_i}{(1-t_i)^2}$ is that they transform integrals (or an homogeneous sum) of the form $[A-B]$ into an homogeneous sum of integrals of the type $[A-1-B]$. The effect of the integration with respect to $\mathrm{d}\theta_1, \mathrm{d}\theta_2$, $\mathrm{d}\phi_1$ and $\mathrm{d}\phi_2$ is that they transform integrals (or an homogeneous sum) of the form $[A-B]$ into an homogeneous sum of integrals of the type $[A+1-B]$.

The effect of the integration of the terms in (8) and (9) is explained as follows. By replacing the measures $\frac{\mathrm{d}a_i \mathrm{d}\theta_i}{(1-a_i^2)^2}$ and $\frac{\mathrm{d}t_i \mathrm{d}\phi_i}{(1-t_i^2)^2}$ by, respectively, $\frac{a_i \mathrm{d}a_i \mathrm{d}\theta_i}{(1-a_i^2)^2}$ and $\frac{t_i \mathrm{d}t_i \mathrm{d}\phi_i}{(1-t_i^2)^2}$ we don't change the uniform convergence of the integrals. The measures are now, for each variable, the measure $\mathrm{d}\lambda_0$ on $\mathbb{D}$. The terms to be integrated may be represented at each step as functions of the hyperbolic distance for a convenient choice of the variables and we may apply Lemma 1 in [Pat1] (see also ([El])).

For example, the above mentioned lemma shows that if $|B| < |A|$ then the integral

$$\int_{\mathbb{D}} \frac{(1-|\eta|)^{2r}}{|A-B\overline{\eta}|^{2r}|1-\eta\overline{w}|^{2r}} \mathrm{d}\lambda_0(\eta) = \int_{\mathbb{D}} \frac{(1-|\eta|^2)^{2r}}{|A|^{2r}|1-(B/A)\overline{\eta}|^{2r}|1-\eta\overline{w}|^{2r}} \mathrm{d}\lambda_0(\eta),$$

is dominated, (modulo a constant and for some ϵ as small as we want), by

$$\frac{1}{(|B|^2-|A|^2)^r(1-|w|^2)^r}\Big[\frac{(|B|^2-|A|^2)(1-|w|^2)}{|A-B\overline{w}|^2}\Big]^{r-\epsilon}.$$

To evaluate the sums in (9) (which is majorizing (8)) we have to go through the following process: We start with a term of the form $[A-(A+1)]$. Integration by $\mathrm{d}\theta_2$ gives an homogeneous sum of terms of the form $[A'-A']$.

The integration by $\frac{\mathrm{d}a_2}{(1-a_2)^2}$ will yield an homogeneous sum of terms of the form $[A''-(A''+1)]$. The square will give homogeneous sum of terms of the form $[A''-(A''+2)]$. The recursive integration by the $\frac{\mathrm{d}t_i \mathrm{d}\phi_i}{(1-t_i)^2}$ will yield an homogeneous sum of the type $[A'''-(A'''+2)]$. The square root will give a similar (eventually an infinite convergent sum) homogeneous sum of the type $[A'''-(A'''+1)]$. The integral with respect to $\mathrm{d}\theta_1$ and the last integral with respect to $\frac{\mathrm{d}a_1}{(1-a_1)^2}$ will get us an homogeneous sum of terms of the form $[A^{(4)}-1-A^{(4)}]$ which means simply a multiple of $\frac{1}{1-s}$ (to which lower degree terms in $\frac{1}{1-s}$ are to be added, e.g $\frac{1}{(1-s)^\alpha}$, $\alpha<1$).

The final form of the integral (leaving aside the factor $(1-s)$ and before performing the last integration by the parameter a_1) is an homogeneous sum of terms of the form

$$\int_0^1 \frac{(1-s)^A(1-a_1)^B}{(1-sa_1)^{A+B+2}} \mathrm{d}a_1.$$

These (hypergeometric) integrals have the leading term $\frac{1}{1-s}$ (compare with example 8, page 297 in [WW]). Finally we have to multiply this by $(1-s)$ (which comes from under the square root). Thus the supremumum of the integrals in in (9), after s is finite.

The singularity behavior for the integrals when the parameters are close to 1, may be explained by the similarity of this integrals with the Appell's double hypergeometric functions ([Ex]).

This completes the proof of Lemma 7 and hence the proof of Proposition 6.

We note that by the same method as above, the computation being this time considerably easier, allows us to show that the integral in the Remark after Proposition 6, with an additional factor $(1-s)^{1/2}$ is convergent. Thus we have the following statement, which is shows a certain mean convergence for the sums involved in the determination of Beardon's exponent of convergence. (In fact for group like the free group one may check this statement directly by using the natural length function replacing the logarithm of the hyperbolic distance). This proposition is not needed

for our proof, but it shows why the first estimate we used for the norm $||e^r_{\overline{z},\zeta}||_1$ fails to give convergence of the integral in Lemma 6.

Corollary. *Let* Γ *be a cocompact subgroup of* $SU(1,1)$*. Let* $\gamma_1, \gamma_2, \ldots$ *be an enumeration of* Γ*. Let* $d(z,\zeta)$ *be the square root of the hyperbolic distance between two points* z, ζ *in* $\mathbb{D}$*. Then the following sums converge, uniformly in* $N \in \mathbb{N}$*:*

$$\sum_{\gamma} [\frac{1}{N^2} \sum_{i=1}^{N} d(\gamma 0, \gamma_i 0)^{2r}]^{1/2}, \;\; N \in \mathbb{N}.$$

We now use the result in Proposition 6 to prove the following statement which estimates the uniform norm on the von Neumann algebras in the Berezin quantization.

Theorem 8. *Let* Γ *be a cocompact subgroup of* $PSL(2,\mathbb{R})$*. For* $r > 1$ *let* π_r *be the projective, unitary representation of* $PSL(2,\mathbb{R})$ *(identified with* $SU(1,1)$*) on the Hilbert space* $H_r = H^2(\mathbb{D}, d\lambda_r)$*. Let* A *be a bounded operator on* H_r *commuting with* $\pi_r(\Gamma)$*. Let* $||\cdot||_{\lambda,r}$ *be the norm (initially defined on a weakly dense subalgebra of the commutant* $\mathcal{A}_r = \{\pi_r(\Gamma)\}'$*. The formula for* $||\cdot||_{\lambda,r}$ *is*

$$||A||_{\lambda,r} = max\ \{sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |\hat{A}(\overline{z},\zeta)|(d(z,\zeta))^r d\lambda_0(\zeta), sup_{\zeta\in\mathbb{D}} \int_{\mathbb{D}} |\hat{A}(\overline{z},\zeta)|(d(z,\zeta))^r d\lambda_0(z)\}.$$

Then there exists a positive constant M_r *and a fixed* $r_0 > 0$ *so that for any* $r > r_0$ *and for all* A *in* $\mathcal{A}_r$ *we have that* $||A||_{\lambda,r}$ *is finite and*

$$||A||_{\infty,r} \leq ||A||_{\lambda,r} \leq M_r ||A||_{\infty,r}.$$

Moreover, keeping the symbol $\hat{A}$ *fixed, but varying* r *in a bounded interval, the constant* M_r *remains bounded.*

Proof. We only have to apply Proposition 6 (and its symmetric version when the rôles of z and ζ are switched). The constant M_r is defined by

$$\max[\sup(\int ||e^r_{\overline{z},\zeta}||_{L^1(\mathcal{A}_r,\tau)} (d(z,\zeta))^r \mathrm{d}\lambda_r(\zeta)), \sup(\int ||e^r_{\overline{z},\zeta}||_{L^1(\mathcal{A}_r,\tau)} (d(z,\zeta))^r \mathrm{d}\lambda_r(z))].$$

Then, from the definition of the norm $||\cdot||_{\lambda,r}$, we deduce that for any A in $\mathcal{A}_r$, one has that for all z in $\mathbb{D}$

$$\int_{\mathbb{D}} |A(\overline{z},\zeta)|(d(z,\zeta))^r \mathrm{d}\lambda_r(\zeta) = \int_{\mathbb{D}} |\tau_{\mathcal{A}_r}(Ae^{r}_{\overline{z},\zeta})|(d(z,\zeta))^r \mathrm{d}\lambda_r(\zeta)$$

$$\leq ||A||_{\infty,r} \int_{\mathbb{D}} ||e^{r}_{\overline{z},\zeta}||_{L^1(\mathcal{A}_r,\tau)}(d(z,\zeta))^r \mathrm{d}\lambda_r(\zeta).$$

This ends the proof

Theorem 9. *Let Γ be a cocompact discrete subgroup of $PSL(2,\mathbb{R})$. For $r>1$ let π_r be the projective unitary representation of $PSL(2,\mathbb{R})$ (identified with $SU(1,1)$) on the Hilbert space $H_r = H^2(\mathbb{D}, d\lambda_r)$. Let $\{\pi_r(\Gamma)\}'$ be the commutant of $\pi_r(\Gamma)$ in $B(H_r)$.*

Note that the type II_1 factor $\{\pi_r(\Gamma)\}'$ is the von Neumann algebras associated to the Berezin's deformation quantization product $*_h$ on functions on $\mathbb{D}/\Gamma$, when $h = 1/r$ and the trace is the integration over a fundamental domain of Γ in $\mathbb{D}$.

Then, for any r, bigger than a fixed r_0, the type II_1 factors $\{\pi_r(\Gamma)\}'$ are mutually isomorphic.

Proof. Indeed in [Ra2] we proved that, (for any fuchsian group Γ), the cyclic, two cocycle ψ_r associated with the deformation ([CFS],[NT], [Ra2]) has the following property

$$\psi_r(A,B,C) \leq c_r ||A||_{\lambda,r}||B||_2||C||_2$$

for all A,B,C in $\{\pi_r(\Gamma)\}'$. Moreover, the constants c_r may be chosen uniformly bounded for r in a bounded interval. Also, in [Ra2] we proved that if one may replace in the above estimate the norm $||\cdot||_{\lambda,r}$ with the uniform norm $||\cdot||_{\infty,r}$ on $B(H_r)$, then the cocycle ψ_r is a coboundary (on the von Neumann algebra). In this case the evolution operator associated with the operator whose coboundary is ψ_r (by considering the operator as a quadratic form), will implement (by [Ra2]) an isomorphism between the algebras $\{\pi_r(\Gamma)\}'$. The preceding statement completes thus the proof for a cocompact group Γ.

Corollary 10. *Let Γ be a cocompact, discrete subgroup of $PSL(2,\mathbb{R})$. Let $\mathcal{L}(\Gamma)$ be the type II_1 factor associated with Γ. Then the fundamental group $\mathcal{F}(\mathcal{L}(\Gamma))$*

contains the rational numbers. Equivalently the algebras $\mathcal{L}(\Gamma)\otimes M_n(\mathbb{C})$ are mutually isomorphic.

Proof. This follows (by the preceding statement) from the computation in ([AS], [Co3], [GHJ]) that $\{\pi_r(\Gamma)\}'$ is isomorphic to $\mathcal{L}(\Gamma)_{[(r-1)(\mathrm{cov}\ \Gamma)/\pi]}$ if $r \geq 2$ is an integer (note that if r is not an integer, this last isomorphism will also hold ([Ra2]) if the group cohomology element in $H^2(\Gamma, \mathbb{T})$ associated with the projective, unitary representation $\pi_r|_\Gamma$ vanishes).

Consequently, we obtain that the algebra $\mathcal{L}(\Gamma)_{[(n-1)(\mathrm{cov}\ \Gamma)/\pi]}$ is isomorphic to the algebra $\mathcal{L}(\Gamma)_{[(m-1)(\mathrm{cov}\ \Gamma)/\pi]}$, for all sufficiently big integers n, m. The result then follows from the fact that $\mathcal{F}(\mathcal{L}(\Gamma))$ is a multiplicative group. This completes the proof.

The following observation is related to the method using in proving Lemma 3. Although this is not related to the subject of this paper we mention it here as a consequence of the method used in this paper. It shows that one may generalize Toeplitz operators with Γ-invariant symbol to Toeplitz operators whose symbol is a finite Γ- invariant measure on $\mathbb{D}/\Gamma$. This operators are also bounded and commute with $\pi_r(\Gamma)$.

Recall that if ϕ is a bounded Γ-invariant function on $\mathbb{D}$, then the corresponding Toeplitz operator T^r_ϕ with symbol ϕ is the compression to $H_r = H^2(\mathbb{D}, \lambda_r)$ of the operator of multiplication with ϕ. Clearly T^r_ϕ commutes with $\pi_r(\Gamma)$, so in the terminology we used in this paper, $T^r_\phi \in \{\pi_r(\Gamma)\}'$.

In particular, the next result shows that there is no positive constant c so that

$$c||T^r_\phi|| = ||T^r_\phi||_{\infty,r} \geq ||\phi||_\infty,$$

for all Γ-invariant bounded measurable functions ϕ on $\mathbb{D}$. If one drops the condition of Γ-invariance this was known to Sarason ([Sar]).

This statement is true because if the above inequality would hold for some constant c then it would follow that any element in $\{\pi_r(\Gamma)\}'$ would be a Toeplitz operator with Γ-invariant, bounded measurable symbol. That these operator do not exhaust all of $\{\pi_r(\Gamma)\}'$ is the content of the next proposition.

Observation. *Let Γ be a cocompact subgroup of $PSL(2,\mathbb{R})$ (identified with $SU(1,1)$)*

finite measure on F. We identify ν with a Γ-invariant measure $\tilde{\nu}$ on $\mathbb{D}$ (which is no longer a finite measure). Consider the quadratic form (eventually unbounded) $< \cdot, \cdot >_\nu$ defined on $H^2(\mathbb{D}, \lambda_r)$ by

$$< f, f >_\nu = \int_{\mathbb{D}} |f|^2 d\tilde{\nu}(z).$$

Then the quadratic form $< \cdot, \cdot >_\nu$ is bounded and defines by a bounded operator T_ν^r in $\{\pi_r(\Gamma)\}'$ of uniform norm less than a universal constant (depending on Γ and r) times the norm $|\nu|(F)$ ([Ru]) of the measure ν:

$$||T_\nu^r||_{\infty,r} \leq const_{r,\Gamma} |\nu|(F).$$

Note that if $d\nu = \phi d\lambda_0$, for a bounded, Γ-invariant function ϕ than the operator T_ν^r corresponding to $< \cdot, \cdot >_\nu$ is T_ϕ^r.

Proof. With the notations in Lemma 3, the (Berezin's) symbol $\hat{A}$ corresponding to the quadratic form $< \cdot, \cdot >_\nu$ is computed by the formula

$$\hat{A}(\overline{z}, \zeta) = \int_{\mathbb{D}} e^r_{\overline{z},\zeta}(\eta, \eta) \mathrm{d}\nu(\eta).$$

To check that this is a bounded operator is sufficient (by ([Ra2]) to show that the norm $||A||_{\lambda,r}$ is finite. Thus we have to estimate

$$\int_{\mathbb{D}} |\hat{A}(\overline{z}, \zeta)| (d(z, \zeta))^r \mathrm{d}\lambda_0(z)$$

$$\leq \int_{\mathbb{D}} [\int_F \sum_\gamma \frac{(1 - |\gamma\eta|^2)^r (1 - \overline{z}\zeta)^r}{(1 - \overline{z}\gamma\eta)^r (1 - \overline{\gamma\eta}\zeta)^r} \mathrm{d}|\nu(\eta)|] (d(z, \zeta))^r \mathrm{d}\lambda_0(\zeta) =$$

$$= \int_F [\sum_\gamma \frac{(1 - |\gamma\eta|^2)^r (1 - |z|^2)^{r/2}}{|1 - (\gamma\eta)\overline{z}|^r} \int_{\mathbb{D}} \frac{1}{|1 - (\overline{\gamma\eta})\zeta|^r} \mathrm{d}\lambda_{r/2}(\zeta)] \mathrm{d}\nu(\eta)$$

$$\leq \int_F \sum_\gamma (d(z, \gamma\eta))^r \mathrm{d}\nu(\eta) = \int_F K_r(z, \eta) \mathrm{d}\nu(\eta).$$

By the estimates in [Le] this quantity is uniformly bounded in z, if r is bigger than twice the exponent of convergence of Γ.

[AO] C. Akemann and P. Ostrand, Computing norms in group $C^a st$-algebras, *Amer. J. Math.*, **98**, (1976), 1015-1047.

[AS] M.F. Atiyah, W. Schmidt, A geometric construction of the discrete series for semisimple Lie groups, *Invent. Math.*, **42**, (1977), 1-62.

[Be] A. F. Beardon, The exponent of convergence of Poincaré series, *Proc. London Math. Soc.*,**18**, (1968),461-483.

[Be] F. A. Berezin, General concept of quantization, *Comm. Math. Phys.*, **40** (1975), 153-174.

[BS] M. Sh. Birman, M. Z. Solomyak, Estimates for singular numbers of integral operators, *Vestnik Leningrad University*, **24**, (1969), 21-28.

[Cob] C. A. Berger, L. A. Coburn, Heat Flow and Berezin-Toeplitz estimates, *American Journal of Math.*, **110**, (1994).

[BMS] M. Bordemann, E. Meirenken, M. Schlicenmaier, Toeplitz quantization of Kaehler manifolds and $gl(N), N \to \infty$ limits, *Mannheimer Manuskripte*, **147**, 1993.

[Co1] A. Connes, Non-commutative Differential Geometry, *Publ. Math., Inst. Hautes Etud. Sci.*, **62**, (1986), 94-144.

[Co2] A. Connes, Sur la Theorie Non Commutative de l'Integration, Algebres d'Operateurs, *Lecture Notes in Math.*, **725**, Springer Verlag.

[Co3] A. Connes, Un facteur du type II_1 avec le groupe fondamentale denombrable, *J. Operator Theory*, **4**, (1980), 151-153.

[CM] A. Connes, H. Moscovici, Cyclic Cohomology, the Novikov Conjecture and Hyperbolic Groups, *Topology*,**29**, (1990), 345-388.

[CS] A. Connes, D. Sullivan, Quantized calculus on S^1 and quasi-Fuchsian groups.

[CFS] A. Connes, M. Flato, D. Sternheimer, Closed star products and Cyclic Cohomology, *Letters in Math. Physics*, **24**, (1992), 1-12.

[CES] E. Christensen, E. G. Effross, A. M. Sinclair, Completely Bounded Multilinear Maps and C^*-algebraic cohomology, *Invent. Math.*, **90**, (1987), 279-296.

[Dyk] K. Dykema, Free products of hyperfinite von Neumann algebras and free dimension, *Duke Math. J.*,**69**,(1993), 97-119.

[El] Elstrodt, J., Die Resolvente zum Eigenwertproblem der automorphen Formen in der hyperbolischen Ebene, I, *Math. Ann.*, **208**, (1974),295-330.

[En] M. Englis, Asymptotics of the Berezin Transform and Quantization on Planar Domains, *Duke Math. Journal*, **79**, (1995) 57-76.

[EM] Eskin, C. McMullen, *Duke Math. Journal*, bf 71, (1994),

[Ex] H. Exton, Handbook of hypergeometric integrals, Ellis Horwood Limited, New York, (1978).

[Gh] J. Barge, E. Ghys, Cocycles d'Euler et de Maslov, *Math. Ann.*, **294**, (1992), 235-265.

[Gr] M. Gromov, Volume and bounded cohomology, *Publ. Math., Inst. Hautes Etud. Sci.*, **56**, (1982), pp. 5-100.

[GHJ] F. Goodman, P. de la Harpe, V.F.R. Jones, Coxeter Graphs and Towers of Algebras, *Springer Verlag, New York, Berlin, Heidelberg*, 1989.

[HP] U. Haagerup, G. Pisier, Bounded linear operators between C^*-algebras, preprint (1993).

[Ha] U. Haagerup, An example of a non nuclear C*-algebra which has the matrix approximation property, *Inventiones Math*, **50**, (1979), 279-293.

[HV] P. de la Harpe, D. Voiculescu, A problem on the type II_1 factors of fuchsian groups, Preprint.

[Hu] H. Huber, Zur analytischen Theorie hyperbolischen Raumformen und Bewegungsgruppen, *Math. Ann.*, **138**, (1959), 1-26.

[Jo] P. Jolissaint, Sur la C* algèbre ré duite de certains groupes discrets d'isométries hyperboliques, *C. R. Acad. Sci. Paris*, **302**, (1986), 657-661.

[Ka] R. V. Kadison, Open Problems in Operator Algebras, Baton Rouge Conference, 1960, mimeographed notes.

[KL] S. Klimek, A.Leszniewski, *Letters in Math. Phys.*, **24**, (1992), pp. 125-139.

[Le] J. Lehner, Automorphic forms, in Discrete groups and automorphic forms, editor J. Harvey, 73-119.

[MvN] F. J. Murray, J. von Neumann, On ring of Operators,IV, *Annals of Mathematics*, **44**, (1943), 716-808.

[NT] R. Nest, B. Tsygan, Algebraic index theorem for families, *Preprint Series, Kobenhavns Universitet*, **28**, (1993).

[Pa1] S. J. Patterson, The Exponent of Convergence of Poincaré series, *Monatshefte fur Mathematik*, **82**, (1976), 297-315.

[Pa2] Spectral Theory and Fuchsian groups, *Math. Proc. Camb. Phil. Soc.*, **81**, (1977), 59-75.

[Pi] Quadratic forms in unitary operators, Preprint 1995.

[Po1] S. Popa, Asymptotic freeness, Preprint 1994.

[PR] S. Popa, F. Rădulescu Derivations of von Neumann Algebras into the Compact Ideal Space of a Semifinite Algebra, *Duke Mathematical Journal*,**57**, (1988), 485-518.

[Ra1] F. Rădulescu, Random matrices, amalgamated free products and subfactors in free group factors of noninteger index, *Inv. Math.* **115**, (1994), 347-389.

[Ra2] F. Rădulescu,Γ- invariant form of the Berezin quantization of the upper half plane, preprint, Iowa 1995.

[Ra3] F, Rădulescu,On the von Neumann Algebra of Toeplitz Operators with Automorphic Symbol, in Subfactors, Proceedings of the Taniguchi Symposium on Operator Algebras, edts. H. Araki, Y. Kawahigashi, H. Kosaki, *World Scientific, Singapore-New Jersey*,1994.

[Ra4] F. Rădulescu, The fundamental group of the von Neumann algebra of a free group with infinitely many generators is $\mathbb{R}_+\setminus\{0\}$, *Journal of the American Mathematical Society*, **5**, (1992), 517-532.

[Ri] M. A. Rieffel, Deformation Quantization and Operator Algebras, *Proc. Symp. Pure Math.*, **51**, (1990), 411-423.

[Ru] W. Rudin, Real and Complex Analysis, Addison -Wesley.

[Sak] Sakai, C^* and W^* Algebras, *Springer Verlag, Berlin Heidelberg New York*, 1964.

[Sal] P. Sally, Analytic Continuation of the Irreducible Unitary Representations of the Universal Covering Group,*Memoirs A. M. S.* , (1968).

[Sar] D. Sarason, personal communication.

[Se] A. Selberg, Harmonic Analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series.*J. Indian Math. Soc. 20*, (1956), 47-87.

[Si] B. Simon, Trace class ideals and their applications, *London Math. Lecture Notes Series*, **35**, 1977.

[SincS] A. M. Sinclair, R. Smith, Hochschild cohomology of von Neumann algebras, *London Math. Soc. Lecture Notes*, **203**, (1995).

[St] W. F. Stinespring, A sufficient condition for an integral operator to have a trace, *Journal Reine Angew. Math.*, **250**, (1958), 200-207.

[To] D. Topping, Lectures on von Neumann Algebras, *Van Nonstrand*, London, (1972)

[Ts] M. Tsuji,Potential Theory in Modern Function Theory, *Maruzen*, Tokyo, 1959.

[Vo1] D. Voiculescu, Entropy and Fisher's information measure in free probability, II, *Inv. Math.*, **118**, (1994), 420-443.

[Vo2] D. Voiculescu, Circular and semicircular systems and free product factors. In Operator Algebras, Unitary Representations, Enveloping algebras and Invariant Theory. *Prog. Math. Boston, Birkhauser*,**92**,(1990), 45-60.

[Vo3] D. Voiculescu, Multiplication of certain non- commuting random variables, *J. Operator Theory*, **18**, (1987), 223-235.

[WW] E. T. Whittaker, G. N. Watson, A course in Modern Analysis, *Cambridge University Press*, (1984).

[We] A. Weinstein, Lecture at the Bourbaki Seminar, Paris, June 1994.